Final Exam Study Guide–M1910 Fall 2012 SOLUTIONS

1. The function \( f(x) = \sqrt[3]{4x - 6} (x + 5)^4 \) has a derivative \( f'(x) = \frac{52(x + 5)^3(x - 1)}{3(4x - 6)^{2/3}} \) so the critical numbers of \( f(x) \) are \( x = -5, \ x = 1 \) and \( x = 3/2 \). The function \( f(x) \) is increasing on \((-\infty, -5) \cup (1, 3/2) \cup (3/2, \infty)\).

2. The function \( R(x) = \frac{2x}{3x^2 + 20} \) has a derivative \( R'(x) = \frac{-2(x^2 - 4)}{5(x^2 + 4)^2} \) and is increasing on \((-2, 2)\). The horizontal asymptote is \( y = 0 \).

3. If \( G(x) = (2x - 1)e^{2x} \) then \( G'(x) = 4xe^{2x} \) and \( G''(x) = 4(2x + 1)e^{2x} \). The function \( G \) is concave upwards on \((-1/2, \infty)\).

4. \( f'(x) = \frac{2}{3}x^{-1/3}(2x - 5) + 2x^{2/3} = \frac{2(2x - 5)}{3x^{1/3}} + \frac{6x}{3x^{1/3}} = \frac{10(x - 1)}{3x^{1/3}} \) so the critical points are \( x = 0, \ 1 \).
5. We consider $T(\theta) = 2 \cos \theta + \sin^2 \theta$ on the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Since $T'(\theta) = -2 \sin \theta (1 - \cos \theta)$ the critical numbers occur wherever $\sin \theta = 0$ or $1 = \cos \theta$, namely $\theta = \pi$. We compare $T\left(\frac{\pi}{2}\right)$, $T\left(\frac{3\pi}{2}\right)$ and $T(\pi)$. The absolute maximum value on the interval occurs at $T\left(\frac{\pi}{2}\right) = T\left(\frac{3\pi}{2}\right) = 1$. The absolute minimum value on the interval occurs at $T(\pi) = -2$.

6. Because $g'(x) = \frac{2x}{x^2 + 1}$, the function is increasing on $(0, \infty)$.

7. Set $R'(t) = \frac{(1-t^2)}{(t^2+1)^2} = 0$ and solve. The critical points are $t = \pm 1$. Check the values $R(-4)$, $R(-1)$ and $R(0)$. The absolute maximum output is $R(0) = 0$. The absolute minimum output is $R(-1) = -\frac{1}{2}$.

8. If $f(x) = 27 + 8x^3 - x^5$, then $f'(x) = 24x^2 - 5x^4$ and $f''(x) = 48x - 20x^3 = 4x(12 - 5x^2)$. The possible points of inflection occur when $f''(x) = 0$, namely $x = \pm \sqrt{\frac{12}{5}} = \pm \frac{2\sqrt{15}}{5}$ and $x = 0$. $f$ is concave down on $(-2\sqrt{15}/5, 0) \cup (2\sqrt{15}/5, \infty)$.

9. $g'(x) = (6x - 12x^2)e^{-4x}$. The critical points occur whenever $6x - 12x^2 = 6x(1 - 2x) = 0$, namely at $x = \frac{1}{2}$ and $x = 0$. Because $g'(x) > 0$ only on the interval $(0, 1/2)$, the function is increasing on $(0, 1/2)$ and decreasing elsewhere. Thus $x = \frac{1}{2}$ is a local maximum and $x = 0$ is a local minimum.
10. Because $G'(x) = 2(x - 4)(x - 7) + (x - 4)^2 = (x - 4)(2x - 14 + x - 4)$ or simplified
$G'(x) = 3(x - 4)(x - 6)$, the critical numbers are $x = 4, 6$.

11. $f'(x) = 1 \cdot \sqrt{9 - 2x} + x \cdot (1/2)(9 - 2x)^{-1/2}(-2)$. To be able to find the zeros and discontinuities of the derivative, we need to find a common denominator.

\[
f'(x) = \frac{\sqrt{9 - 2x}}{1} + \frac{-2x}{2\sqrt{9 - 2x}} = \frac{9 - 2x}{\sqrt{9 - 2x}} + \frac{-x}{(9 - 2x)^{1/2}} = \frac{9 - 3x}{\sqrt{9 - 2x}}.
\]

The domain of $f$ is $(-\infty, \frac{9}{2})$ with a critical number at $x = 3$.

12. $T'(x) = \cos x - \sin x$ and $T''(x) = -\sin x - \cos x$. To find the possible points of inflection (PPOI), we set $T''$ equal to zero and solve the resulting equation: $\sin x = -\cos x$. Thus, the only inflection point in the interval $(0, \pi)$ is $x = \frac{3\pi}{4}$. HINT: Think about the unit circle. The central angle $\frac{3\pi}{4}$ radians is associated with the ordered pair $\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Function $T$ is concave up on the interval $\left(\frac{3\pi}{4}, \pi\right)$.

13. a. The x-coordinate where the projectile strikes the ground is a positive value of $x$ such that $x - \frac{32}{16}x^2 = 0$, or $x = \frac{\sqrt{\frac{\pi}{2}}}{8}$

b. $y' = 0$ when the projectile reaches its maximum height

c. We solve $y' = 0$ and find that when $x = \frac{\sqrt{\frac{\pi}{2}}}{8}$, the projectile reaches its maximum height of $y = \frac{\sqrt{\frac{\pi}{2}}}{128}$. 


14. \( f'(x) = \frac{-20(x + 3)^3}{(x - 2)^5} \)

15. \( g'(x) = \frac{2(2x + 1)^2(7 - 4x)}{(1 - 4x)^3} \)

16. \( h'(x) = \frac{-6x}{\sqrt{1 - 9x^4}} \)

17. \( T'(x) = \frac{6}{(x - 8)^{1/2}(x + 4)^{3/2}} \)

18. \( f'(y) = (3y^2e^{5y} + 5y^3e^{5y}) \sin y + y^3e^{5y} \cos y \)

\[ = y^3e^{5y}(3 \sin y + 5y \sin y + y \cos y) \]

19. \( g'(x) = 12(x^7 - 4x^2)^2(7x^6 - 8x) \)

\[ = 12x^5(x^5 - 4)^2(7x^6 - 8) \]

20. \( h'(t) = -2t \sin t^2 - 1 \)

21. \( T'(w) = 2w (w^2 + w - 30) + (w^2 - 9)(2w + 1) \)

\[ = 4w^3 + 3w^2 - 78w - 9 \]

22. \( R'(\theta) = \frac{\sqrt{7}(1 - \theta) \sec^2(\theta) + \sqrt{7} \tan \theta}{(1 - \theta)^2} \)

23. \( W'(t) = \frac{\pi \cos \pi t}{2\sqrt{\sin \pi t}} \)

24. If we define \( x \) to be the length of side of the square cut from each corner, then the height of the resulting open box will be \( x \). The volume of the open box can be written as a function of \( x \): \( V(x) = x(16 - 2x)(16 - 2x) = x(16 - 2x)^2 \).

To maximize the volume, we find the critical numbers of \( V \). The derivative of \( V \) is \( V'(x) = (16 - 2x)^2 - 4x(16 - 2x) = 4(3x - 8)(x - 8) \). It can be shown that \( x = 8/3 \) is a relative maximum of the function \( V \), therefore the dimension of the box are \( \frac{8}{3} \times \frac{32}{3} \times \frac{32}{3} \) and the maximum volume is \( \frac{8192}{27} \approx 303.407 \) cubic inches.
25. If we define \( x \) to be the length of the side of the square base, then the volume of the closed rectangular container would be \( V = x^2 h \). We are given that the container should hold 2000 cubic inches, thus \( h = \frac{2000}{x^2} \).

The box will have a top and a bottom each costing $2 per square inch. The box will have four side panels each costing $3 per square inch. Thus the total cost of materials for the container is

\[
C(x) = 2(2)x^2 + 4(3)xh = 4x^2 + 12x \left( \frac{2000}{x^2} \right)
\]

To minimize the cost, we find the critical numbers of \( C \). The derivative of \( C \) is

\[
C'(x) = 8x - \frac{48000}{x^3}
\]

It can be shown that \( x = 2\sqrt[3]{375} \approx 8.801 \) is a relative minimum of the function \( C \), therefore the dimension of the box are \( 2\sqrt[3]{375} \times 2\sqrt[3]{375} \times 20\sqrt[3]{3} \).

26. \(-\frac{1}{3}x^{-3} + C\)  
27. \(2x^4 + x^3 + \pi x + C\)  
28. \(x - 2 \cos x + C\)  
29. \(\frac{1}{15}(x^3 + 5)^5 + C\)  
30. \(\frac{1}{6}(x^4 + 6)^{3/2} + C\)  
31. \(-\frac{1}{4(1+x^4)} + C\)  
32. \(-\frac{\cos(\pi x)}{\pi} + C\)  
33. \(-\frac{1}{3}e^{-x^3} + C\)  
34. \(\frac{1}{4}x^4 - 4x^2 + C\)  
35. \(\frac{1}{4}(\sin(2x))^2 + C\)  
36. \(\frac{3}{2}x^2|_2 = 48\)  
37. \(\frac{2}{5}x^{5/2}|_1 = \frac{62}{5}\)  
38. 2  
39. 9  
40. \(\tan(\pi/6) - \tan(-\pi/6) = \frac{2\sqrt{3}}{3}\)  
41. \(4\sec(\pi/3) - 4\sec(-\pi/3) = 0\)  
42. \(F'(x) = \frac{d}{dx} \left[ \int_{-3}^{5x^2} 2t^3 - 7t \, dt \right] = 10x \left[ (5x^2)^3 - 7(5x^2) \right] = 2500x^7 - 350x^3\)

43. Since direct substitution of \( x = 2 \) would yield \( 2/0 \) which is not one of the indefinite forms, \( \lim_{x \to 2} \frac{5 - \sqrt{7} + x}{x - 2} \) does not exist.
44. Direct substitution of \( x = 1 \) yields the indefinite form \( 0/0 \), so we use L'Hopital’s Rule to re-write the limit until we can use Direct Substitution to evaluate the limit:

\[
\lim_{x \to 1} \frac{x^3 + x^2 - 2x}{x - 1} = \lim_{x \to 1} \frac{3x^2 + 2x - 2}{1} = 3
\]

45. \( \lim_{x \to 0} \frac{\ln 3x}{x} \) does not exist.

46. This limit is of indefinite form \( \infty/\infty \), so we use L'Hopital’s Rule to re-write the limit until we can evaluate the limit.

\[
\lim_{x \to \infty} \frac{3 - 2x}{x + 7} = \lim_{x \to \infty} \frac{-2}{1} = -2.
\]

47. This limit is of indefinite form \( \infty/\infty \), so we use L'Hopital’s Rule to re-write the limit. We will apply LH three times before we can evaluate the limit.

\[
\lim_{x \to \infty} \frac{x^3 - 3x^2 + 4}{x^4 - 4x^3 + 7x^2 - 12x + 12} = \lim_{x \to \infty} \frac{3x^2 - 6x}{4x^3 - 12x^2 + 14x - 12} \\
= \lim_{x \to \infty} \frac{6x - 6}{12x^2 - 24x + 14} = \lim_{x \to \infty} \frac{6}{24x - 24} = 0
\]

48. Direct substitution of \( t = 0 \) yields the indefinite form \( 0/0 \), so we use L'Hopital’s Rule to re-write the limit until we can use Direct Substitution to evaluate the limit:

\[
\lim_{t \to 0} \frac{2e^t - 2}{t} = \lim_{t \to 0} \frac{2e^t}{1} = 2
\]

49. Direct substitution of \( t = 0 \) yields the indefinite form \( 0/0 \), so we use L'Hopital’s Rule to re-write the limit until we can use Direct Substitution to evaluate the limit. Recall, the derivative of the exponential function base \( a \):

\[
\frac{d}{dt} [a^t] = (\ln a)a^t
\]

\[
\lim_{t \to 0} \frac{4^t - 6^t}{\sin 7t} = \lim_{t \to 0} \frac{(\ln 4)4^t - (\ln 6)6^t}{7 \cos 7t} = \ln \frac{4 - \ln 6}{7}
\]

50. Direct substitution of \( x = 0 \) yields the indefinite form \( 0/0 \), so we use L'Hopital’s Rule to re-write the limit until we can use Direct Substitution to evaluate the limit:

\[
\lim_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}
\]
51. Direct substitution of \( w = 0 \) yields the indefinite form \( 0/0 \), so we use L'Hopital’s Rule:

\[
\lim_{w \to 0} \frac{2 \tan^{-1}(w)}{w} = \lim_{w \to 0} \frac{2}{w^2 + 1} = 2
\]

52. As written, this limit is of the form \( \infty \cdot 0 \). We re-write so that a factor of \( e^{x^2} \) is in the denominator. Then the limit is of indefinite form \( \infty / \infty \) and we can use L'Hopital’s Rule to re-write the limit.

\[
\lim_{x \to \infty} x^3 e^{-x^2} = \lim_{x \to \infty} \frac{x^3}{e^{x^2}} = \lim_{x \to \infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x \to \infty} \frac{3x}{2e^{x^2}} = \lim_{x \to \infty} \frac{3}{4xe^{x^2}} = 0
\]

53. As written, this limit is of the form \( 1^\infty \). We will need to introduce a natural logarithm as follows:

\[
y = \lim_{w \to \infty} \left( 1 + \frac{1}{x} \right)^x
\]

\[
\ln y = \ln \left[ \lim_{w \to \infty} \left( 1 + \frac{1}{x} \right)^x \right]
\]

\[
\ln y = \lim_{w \to \infty} \left[ \ln \left( 1 + \frac{1}{x} \right)^x \right]
\]

\[
\ln y = \lim_{w \to \infty} \left[ x \cdot \ln \left( 1 + \frac{1}{x} \right) \right]
\]

Now as written, this limit is of the form \( \infty \cdot 0 \). We re-write so that a factor of \( 1/x \) is in the denominator. Then the limit is of indefinite form \( 0/0 \) and we can use L'Hopital’s Rule to re-write the limit.

\[
\ln y = \lim_{w \to \infty} \left[ \frac{\ln (1 + \frac{1}{x})}{\frac{1}{x}} \right]
\]

\[
\ln y = \lim_{w \to \infty} \left[ \frac{\frac{d}{dx} \left( 1 + \frac{1}{x} \right)}{\frac{1}{x}} \right]
\]

\[
\ln y = \lim_{w \to \infty} \left[ \frac{\frac{x}{(1 + \frac{1}{x})(-1)}}{x^2} \right]
\]

\[
\ln y = \lim_{w \to \infty} \left[ \frac{1}{1 + \frac{1}{x}} \right]
\]

\[
\ln y = 1
\]

\[
y = e
\]

Therefore, \( y = \lim_{w \to \infty} \left( 1 + \frac{1}{x} \right)^x = e \)
54. We must check the two conditions of the MVT. Is \( f(x) = x^3 \) continuous on the closed interval \([0, 1]\)? And is \( f(x) = x^3 \) differentiable on the open interval \((0, 1)\)?

Since \( f \) is a polynomial it is continuous and differentiable on its entire domain. Thus, YES, the Mean Value Theorem applies.

Now, find all the values \( c \) in the open interval \((0, 1)\) such that \( f'(c) = \frac{f(1) - f(0)}{1 - 0} \).

The derivative of \( f \) is \( f'(x) = 3x^2 \) and the slope of the secant line from \( a = 0 \) to \( b = 1 \) is \( \frac{f(1) - f(0)}{1 - 0} = \frac{1^3 - 0^3}{1 - 0} = 1 \).

Solving the resulting equation: \( 3x^2 = 1 \) we observe that \( x = \pm \frac{\sqrt{3}}{3} \), but only the positive value is in the given interval. Therefore, \( c = \frac{\sqrt{3}}{3} \).

55. We must check the two conditions of the MVT. Is \( f(x) = x^4 - 8x \) continuous on the closed interval \([0, 2]\)? And is \( f(x) = x^4 - 8x \) differentiable on the open interval \((0, 2)\)?

Since \( f \) is a polynomial it is continuous and differentiable on its entire domain. Thus, YES, the Mean Value Theorem applies.

Now, find all the values \( c \) in the open interval \((0, 2)\) such that \( f'(c) = \frac{f(2) - f(0)}{2 - 0} \).

The derivative of \( f \) is \( f'(x) = 4x^3 - 8 \) and the slope of the secant line from \( a = 0 \) to \( b = 2 \) is \( \frac{f(2) - f(0)}{2 - 0} = \frac{(16 - 16) - (0 - 0)}{2 - 0} = 0 \).

Solving the resulting equation: \( 4x^3 - 8 = 0 \) we observe that \( x = \sqrt[3]{2} \).

56. We must check the two conditions of the MVT. Is \( f(x) = \frac{x+1}{x} \) continuous on the closed interval \([-1, 3]\)? And is \( f(x) = \frac{x+1}{x} \) differentiable on the open interval \((-1, 3)\)?

Immediately we can see that \( f \) is not continuous at \( x = 0 \), so, NO, the MVT does not apply.

57. Since \( f(x) \) is continuous on \([0, 2]\) and the derivative \( f(x) = \frac{8}{(10 + 4x)^{1/3}} \) exists on \([0, 2]\), YES, the Mean Value Theorem does apply.
58. Since $T(x)$ is continuous on $[-\pi/4, \pi/4]$ and the derivative $g(x) = \sec^2 x$ exists on $[-\pi/4, \pi/4]$, YES, the Mean Value Theorem does apply.

59. We observe that the function $f(x) = \sqrt{x}$ is continuous on the closed interval $[4, 9]$, so the MVT for integrals will apply. First we calculate the definite integral (and do not round the answer):

$$
\int_{4}^{9} x^{1/2} \, dx = \frac{2}{3} x^{3/2}\big|_{4}^{9} = \frac{38}{3}.
$$

Next, we solve the equation $f(c)(b - a) = \int_{a}^{b} f(x) \, dx$:

$$5\sqrt{c} = \frac{38}{3}
$$

The value of $c$ guaranteed by the MVT for integrals is $c = \frac{1444}{225} \approx 6.4178$. We note that $4 < c < 9$.

60. We observe that the function $f(x) = \frac{9}{x^3}$ is continuous on the closed interval $[1, 3]$, so the MVT for integrals will apply. First we calculate the definite integral:

$$
\int_{1}^{3} 9x^{-3} \, dx = -\frac{9}{2} x^{-2}\big|_{1}^{3} = 4
$$

Next, we solve the equation $f(c)(b - a) = \int_{a}^{b} f(x) \, dx$:

$$\frac{2 \cdot 9}{x^3} = 4
$$

The value of $c$ guaranteed by the MVT for integrals is $c = \frac{3\sqrt[3]{9}}{2} = \frac{3\sqrt[3]{5}}{2} \approx 1.6510$. We note that $1 < c < 3$. 

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