3.6 Expectation values

We have seen that for a spin-\(\frac{1}{2}\) particles in a state

\[
|\psi\rangle = c_+ |+Z\rangle + c_- |−Z\rangle
\]

The average value for \(S_z\) which we denoted as \(\langle S_z \rangle\) is given by

\[
\langle S_z \rangle = |c_+|^2 \left( +\frac{\hbar}{2} \right) + |c_-|^2 \left( -\frac{\hbar}{2} \right).
\]

Where \(|c_+|^2\) and \(|c_-|^2\) are the probabilities of the particle to be found in the states \(+Z\rangle\) and \(−Z\rangle\), respectively. In the \(S_z\) basis we can express these probabilities as

\[
|c_+|^2 = c_+ c_+^* = \langle +z |\psi\rangle \langle \psi |+Z\rangle,
\]
\[
|c_-|^2 = c_- c_-^* = \langle −z |\psi\rangle \langle \psi |−Z\rangle
\]

so that

\[
\langle S_z \rangle = \langle \psi |+Z\rangle \left( +\frac{\hbar}{2} \right) \langle +z |\psi\rangle 
+ \langle \psi |−Z\rangle \left( -\frac{\hbar}{2} \right) \langle −z |\psi\rangle
\]
or
\[
\langle S_z \rangle = \langle \psi | +Z \rangle \left( +\frac{\hbar}{2} \right) \langle +z | \psi \rangle + \langle \psi | -Z \rangle (0) \langle -z | \psi \rangle \\
+ \langle \psi | +Z \rangle (0) \langle +z | \psi \rangle + \langle \psi | -Z \rangle \left( -\frac{\hbar}{2} \right) \langle -z | \psi \rangle.
\]

The above expression can be expressed in matrix form as
\[
\langle S_z \rangle = \left( \begin{array}{cc} \langle \psi | +Z \rangle & \langle \psi | -Z \rangle \end{array} \right) \\
\times \left( \begin{array}{cc} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{array} \right) \left( \begin{array}{c} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{array} \right).
\]

But we know that
\[
\left( \begin{array}{cc} \langle \psi | +Z \rangle & \langle \psi | -Z \rangle \end{array} \right)
\]

is the matrix representation for \( \langle \psi | \),

\[
\left( \begin{array}{c} \langle +z | \psi \rangle \\ \langle -z | \psi \rangle \end{array} \right)
\]

is the matrix representation for \( | \psi \rangle \), and

\[
\left( \begin{array}{cc} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{array} \right)
\]
is the matrix representation for the operator $\hat{J}_z$ in the $S_z$ basis. Therefore, the expectation value for $\hat{S}_z$ can be expressed as

$$\langle \hat{S}_z \rangle = \langle \psi | \hat{J}_z | \psi \rangle$$

- Expectation value of any physical observable that is can be represented by an operator $\hat{A}$ can be expressed as

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

where the state vector $|\psi\rangle$ can be expressed as in terms of any basis vector.

Ex 1 For the operator $\hat{J}_z$

(a.) Find the matrix representation in the $\hat{S}_z$ basis

(b.) Change the matrix representation of $\hat{J}_z$ from the $\hat{S}_z$ basis to $\hat{S}_y$
we recall that for a state

\[ |\psi\rangle = \sum_n c_n |a_n\rangle \]

an expectation value for a physical observable \( \hat{A} \) is given by

\[ \langle \hat{A} \rangle = \sum_n |c_n|^2 a_n = \sum_n c_n^* c_n a_n \]

where \( a_n \) is the eigenvalue for the observable \( \hat{A} \) satisfying the eigenvalue equation

\( \hat{A} |a_n\rangle = a_n |a_n\rangle \)

we also know that

\[ c_n = \langle a_n | \psi \rangle, \quad c_n^* = \langle \psi | a_n \rangle \]

so that

\[ \langle \hat{A} \rangle = \sum_n \langle \psi | a_n \rangle \langle a_n | \psi \rangle a_n \]

\[ = \sum_n \langle \psi | \hat{A} | a_n \rangle \langle a_n | \psi \rangle \]

\[ \langle \hat{A} \rangle = \sum_n \langle \psi | \hat{a} | a_n \rangle \langle a_n | \psi \rangle \]
which we may write as \( \psi | \hat{A} \geq \sum_n a_n | \lambda_n \rangle | \psi \rangle \)

or

\( \langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \)

such that

\( \sum_n (a_n) | \lambda_n |^2 = 1 \)
3.7 Photon Polarization and the Spin of the photon

Light is an electromagnetic wave. Waves can also behave like particles. *Photons are particles of light whose energy is proportional to the frequency of the light* \((E=\hbar\omega)\) which can be shown by quantization of the electromagnetic wave. We are not going to do that here. The quantum description of light is significant when we deal with small number of photons. What we are going to do here is applying what we have studied so far for spin-\(\frac{1}{2}\) particles to describe the state of polarization of light. The polarization of light is determined by the electric field, \(\vec{E}\). The interaction of light with matter, in most cases, is electrical. So we consider the electric field of the electromagnetic field. We assume this wave is plane wave and propagating along the positive z-direction. We can then express the electric field of this wave as

\[
\vec{E} = (E_{0x}\hat{x} + E_{0y}\hat{y}) e^{i(kz-\omega t)},
\]
where \( k \) is the wave number, \( \omega \) is the angular frequency, \( E_{0x} \) and \( E_{0y} \) are the amplitude of the electric field in the \( x \) and \( y \), direction, respectively, which are complex in general

\[
E_{0x} = |E_{0x}| e^{i\varphi_x}, \quad E_{0y} = |E_{0y}| e^{i\varphi_y}
\]

If we assume the phase of the \( x \) component \( \varphi_x = 0 \) and and that of the \( y \) component \( \varphi_y = \varphi \), as measured relative to the phase of the \( x \) component of the electric field, then we can write

\[
\vec{E} = \left( |E_{0x}| \hat{x} + |E_{0y}| e^{i\varphi} \hat{y} \right) e^{i(kz - \omega t)}
\]

\[
\Rightarrow \quad \vec{E} = |E_{0x}| e^{i(kz - \omega t)} \hat{x} + |E_{0y}| e^{i(kz - \omega t + \varphi)} \hat{y}
\]

- **Linearly Polarized**: If the phase \( \varphi \) is zero, the electric field becomes

\[
\vec{E} = |E_{0x}| e^{i(kz - \omega t)} \hat{x} + |E_{0y}| e^{i(kz - \omega t)} \hat{y}
\]

\[
\Rightarrow \quad \vec{E} = \left( |E_{0x}| \hat{x} + |E_{0y}| \hat{y} \right) e^{i(kz - \omega t)}
\]

and light is linearly polarized. Under this condition if \( |E_{0y}| = 0 \) the light is \( x \)-polarized; and if \( |E_{0x}| = 0 \), the light is \( y \)-polarized.
• **Elliptically polarized**: If the phase difference between the $x$ and $y$ components of the electric field is $\pm \pi/2$ ($\varphi = \pm \pi/2$), the electric field becomes

$$
\vec{E} = |E_{0x}| e^{i(kz-\omega t)} \hat{x} + |E_{0y}| e^{i(kz-\omega t \pm \frac{\pi}{2})} \hat{y}
$$

$$
\Rightarrow \vec{E} = \left( |E_{0x}| \hat{x} \pm i |E_{0y}| \hat{y} \right) e^{i(kz-\omega t)}
$$

• **Circularly polarized**: If the phase difference between the $x$ and $y$ components of the electric field is $\pm \pi/2$ ($\varphi = \pm \pi/2$) and also $|E_{0x}| = |E_{0y}| = E_0$, the electric field becomes

$$
\vec{E} = E_0 e^{i(kz-\omega t)} \hat{x} + E_0 e^{i(kz-\omega t \pm \frac{\pi}{2})} \hat{y}
$$

$$
\Rightarrow \vec{E} = (E_0 \hat{x} \pm i E_0 \hat{y}) e^{i(kz-\omega t)}.
$$

In reality the electric field is given by $\text{Re} \left( \vec{E} \right)$. Therefore, for a circularly polarized light we may write

$$
\text{Re} \left( \vec{E} \right) = \text{Re} \left[ (E_0 \hat{x} \pm i E_0 \hat{y}) \right]
$$

$$
\times (\cos (kz - \omega t) + i \sin (kz - \omega t))
$$

$$
\Rightarrow \text{Re} \left( \vec{E} \right) = \text{Re} \left[ (E_0 \hat{x} \pm i E_0 \hat{y}) \right]
$$

$$
\times (\cos (kz - \omega t) + i \sin (kz - \omega t))
$$

$$
\Rightarrow \text{Re} \left( \vec{E} \right) = E_0 \cos (kz - \omega t) \hat{x} \mp E_0 \hat{y} \sin (kz - \omega t)
$$
For $\varphi = \pi/2$ we find

$$\text{Re} \left( \vec{E} \right) = E_0 \cos (kz - \omega t) \hat{x} - E_0 \hat{y} \sin (kz - \omega t)$$

and $\varphi = -\pi/2$

$$\text{Re} \left( \vec{E} \right) = E_0 \cos (kz - \omega t) \hat{x} + E_0 \hat{y} \sin (kz - \omega t).$$

Now let’s see how the electric field evolves with time at $z = 0$. For $\varphi = \pi/2$

$$\text{Re} \left( \vec{E} \right) = E_0 \cos (-\omega t) \hat{x} - E_0 \hat{y} \sin (-\omega t)$$

$$\Rightarrow \text{Re} \left( \vec{E} \right) = E_0 \cos (\omega t) \hat{x} + E_0 \hat{y} \sin (\omega t)$$

This means that, as shown in the figure below the electric field rotates from the positive $x$-axis to the positive $y$ axis. This means if we curl right hand from the $x$ axis to the $y$ axis our thumb points in the direction of propagation which is the positive axis, and it is known as right-circularly polarized. On the other hand for $\varphi = -\pi/2$

$$\text{Re} \left( \vec{E} \right) = E_0 \cos (-\omega t) \hat{x} + E_0 \hat{y} \sin (-\omega t)$$

$$\Rightarrow \text{Re} \left( \vec{E} \right) = E_0 \cos (\omega t) \hat{x} - E_0 \hat{y} \sin (\omega t)$$
and the electric field rotates from the positive \( x \) axis towards the negative \( y \) axis. This means if we put our left hand and curl it from positive \( x \) to negative \( y \) our thumb points in the direction of propagation of the light, and the light is called \textit{left-circularly polarized} light.

- Now going back to our complex representation of the electric field we can describe the states of polariza-
tion light as

\[ \vec{E} = \left( |E_{0x}| \hat{x} + |E_{0y}| \hat{y} \right) \]
linearly polarized

\[ \vec{E} = \left( |E_{0x}| \hat{x} + i |E_{0y}| \hat{y} \right) \]
right-eliptically polarized

\[ \vec{E} = \left( |E_{0x}| \hat{x} - i |E_{0y}| \hat{y} \right) \]
left-eliptically polarized

\[ \vec{E} = (E_0 \hat{x} + iE_0 \hat{y}) \]
right-circularly polarized

\[ \vec{E} = (E_0 \hat{x} - iE_0 \hat{y}) \]
left-circularly polarized

Let’s consider polarized photons with a unit magnitude of electric field with direction \( \varphi \) as measured from the positive \( x \) axis, and if we represent photons which are \( x \) polarized \((\vec{E} = \hat{x})\) by the ket vector \(|X\rangle\) and photons which are \( y \) polarized \((\vec{E} = \hat{y})\) by \(|Y\rangle\),
then we can write

\[ |LN\rangle = (\cos \varphi |X\rangle + \sin \varphi |Y\rangle) \]

linearly polarized

\[ |RE\rangle = (\cos \varphi |X\rangle + i \sin \varphi |Y\rangle) \]

right-elliptically polarized

\[ |LE\rangle = (\cos \varphi |X\rangle - i \sin \varphi |Y\rangle) \]

left-elliptically polarized

For circularly polarized light since the x and y component of the electric field are the same we must have \( \cos \varphi = \sin \varphi \Rightarrow \varphi = \frac{\pi}{4} \). Which leads to

\[ |RC\rangle = \frac{1}{\sqrt{2}} (|X\rangle + i |Y\rangle) \]

right-circularly polarized

\[ |LC\rangle = \frac{1}{\sqrt{2}} (|X\rangle - i |Y\rangle) \]

left-circularly polarized.

The basis \(|X\rangle\) and \(|Y\rangle\) are orthonormal

\[ \langle X | Y \rangle = \langle Y | X \rangle = 0 \]
since and $y$-polarized photons can not pass through a Polaroid with a transmission axis along the $x$ direction and $x$-polarized photons can not pass through a polaroid with a transmission axis along the $y$ direction::

$$\langle Y | Y \rangle = \langle X | X \rangle = 1$$

for an ideal polaroid with a transmission axis along the $x$-direction can pass all $x$ polarized photons and the same is true for $y$. Therefore the basis $|X\rangle$ and $|Y\rangle$ form a complete set

$$|X\rangle \langle X| + |Y\rangle \langle Y| = 1$$

Ex 3 Suppose the transmission axis of a polaroid represented by the ket vector $|X'\rangle$ is rotated by $\varphi$ counterclockwise as about the $z$-axis as shown in the figure below. The complete absorption direction of the polaroid represented by $|Y'\rangle$ is normal to $|X'\rangle$.  Express these vectors in $|X\rangle$ and $|Y\rangle$ basis.
Sol: Using the completeness relation we can write

\[ |X'\rangle = (|X\rangle \langle X| + |Y\rangle \langle Y|) |X'\rangle \]
\[ |Y'\rangle = (|X\rangle \langle X| + |Y\rangle \langle Y|) |Y'\rangle \]

\[ |X'\rangle = |X\rangle \langle X| X'\rangle + |Y\rangle \langle Y| X'\rangle \]
\[ |Y'\rangle = |X\rangle \langle X| Y'\rangle + |Y\rangle \langle Y| Y'\rangle \]

\[ |X'\rangle = \cos \varphi |X\rangle + \sin \varphi |Y\rangle \]
\[ |Y'\rangle = \cos (90 + \varphi) + \sin (90 + \varphi) |Y\rangle \]
\[ = - \sin \varphi |X\rangle + \cos \varphi |Y\rangle \]
Now based on the example of a right circularly polarized light vector for right circularly polarized light:

$$|A\rangle = \frac{1}{\sqrt{2}} \left[ |x\rangle + i |y\rangle \right]$$

we may write

$$|R\rangle = \frac{1}{\sqrt{2}} \left[ (\cos \alpha |x\rangle + \sin \alpha |y\rangle) 
+ i (-\sin \alpha |x\rangle + \cos \alpha |y\rangle) \right]$$

$$\Rightarrow |R\rangle = \frac{1}{\sqrt{2}} \left[ (\cos \alpha - i \sin \alpha) |x\rangle 
+ (\sin \alpha + i \cos \alpha) |y\rangle \right]$$

$$\Rightarrow |R\rangle = \frac{1}{\sqrt{2}} \left[ e^{-i\alpha} |x\rangle + i (\cos \alpha - i \sin \alpha) |y\rangle \right]$$

$$\Rightarrow |R\rangle = \frac{1}{\sqrt{2}} e^{-i\alpha} (|x\rangle + i |y\rangle)$$

$$\Rightarrow |R\rangle = e^{-i\alpha} |R\rangle$$

This means the rotation of the $|x\rangle$ and $|y\rangle$ by an angle $\alpha$ (CCW adds a
a phase factor. We recall for spin-$\frac{1}{2}$ states, the operator
\[ \hat{R}(\alpha \hat{\lambda}) = e^{i \frac{\alpha}{2} \hat{\lambda}} \]

rotated acting on spin-$\frac{1}{2}$ particle state vector $|\uparrow\downarrow\rangle$

\[ \hat{R}(\alpha \hat{\lambda}) |\uparrow\downarrow\rangle = e^{i \frac{\alpha}{2} \hat{\lambda}} |\uparrow\downarrow\rangle = e^{i \frac{\alpha}{2} \hat{\lambda}} |\uparrow\downarrow\rangle \]

which means it rotates is by $\alpha/2$ sine for

\[ \hat{J}_z |\pm \rangle = \pm \frac{\hbar}{2} |\pm \rangle \]

In the case of the photon
\[ \hat{R}(\alpha \hat{\lambda}) = e^{i \frac{\alpha}{2} \hat{\lambda}} \]

gives
\[ \hat{R}(\alpha \hat{\lambda}) |R\rangle = e^{i \frac{\alpha}{2} \hat{\lambda}} |R\rangle \]

thus we conclude from eigenvalue equation for photon spin
\[ \hat{J}^2 |R\rangle = \hbar |R\rangle \]
\[ \hat{J}^z |L\rangle = -\hbar |L\rangle \]
Example: In the $|L>, |1L>$ basis

\[
\hat{J}_z \rightarrow \begin{bmatrix}
    \langle R | \hat{J}_z | R \rangle & \langle R | \hat{J}_z | L \rangle \\
    \langle L | \hat{J}_z | R \rangle & \langle L | \hat{J}_z | L \rangle
\end{bmatrix}
\]

= \begin{bmatrix}
    + \hat{z} & 0 \\
    0 & -\hat{z}
\end{bmatrix}

In the $|L>, |1L>$ basis, noting that

\[
\langle \hat{R} | \hat{J}_z | \hat{R} \rangle = \langle \hat{R} | (|L> < L| + |R> < R|) | L \rangle
\]

\[
\langle \hat{L} | \hat{J}_z | \hat{L} \rangle = \langle \hat{L} | (|L> < L| + |R> < R|) | L \rangle
\]

\[
\hat{J}_z \langle \hat{R} | \hat{J}_z | \hat{R} \rangle \langle \hat{R} | \hat{J}_z | \hat{L} \rangle \langle \hat{L} | \hat{J}_z | \hat{L} \rangle
\]

which leads to the matrix form

\[
\langle \hat{L} | \hat{J}_z | \hat{L} \rangle = \begin{bmatrix}
    + \hat{z} & \hat{z}_2 \\
    \hat{z}_2 & -\hat{z}
\end{bmatrix}
\]
where
\[
\hat{T} = \begin{pmatrix}
\langle R' | x \rangle & \langle R' | y \rangle \\
\langle L' | x \rangle & \langle L' | y \rangle
\end{pmatrix}
\]

using
\[
| R \rangle = \frac{1}{\sqrt{2}} [ | x \rangle + i | y \rangle ]
\]
and
\[
| L \rangle = \frac{1}{\sqrt{2}} [ | x \rangle - i | y \rangle ]
\]

we find
\[
\hat{T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} = \frac{a + 1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}
\]

so that
\[
\frac{1}{2} \sum_{|x\rangle, |y\rangle} \langle x | \hat{T} | y \rangle [ | x \rangle [k \ 0] | y \rangle
\]
basis
\[
= h \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]
4 Angular Momentum

4.1 Commutation relations and eigenvalues for the rotation operators

4.2 Degeneracy and simultaneous Observable

- *Simultaneous eigenfunctions*: suppose the eigenfunction $u_a(x)$ is the eigenfunction to two different observable $A$ and $B$ described by the operators $\hat{A}$ and $\hat{B}$ such that

\[
\hat{A}u_a(x) = au_a(x), \quad \hat{B}u_a(x) = bu_a(x).
\] (8)

then we have

\[
\hat{B}\hat{A}u_a(x) = a\hat{B}u_a(x) = abu_a(x)
\] (9)

\[
\hat{A}\hat{B}u_a(x) = b\hat{A}u_a(x) = abu_a(x).
\]
so that

\[ (\hat{A}\hat{B} - \hat{B}\hat{A}) u_a (x) = 0. \]  \hspace{1cm} (10)

This result leads to the conclusion that if two operators \(\hat{A}\) and \(\hat{B}\) have simultaneous eigenfunctions, then the two operators must commute

\[ [\hat{A}, \hat{B}] = 0 \]

The converse is also true. That means if two or more operators commute, then they have simultaneous eigenfunctions and the eigenvalue equations for the two commuting operators should be written as

\[ \hat{A}v_{ab} (x) = av_{ab} (x), \hat{B}v_{ab} (x) = bv_{ab} (x). \] \hspace{1cm} (11)

We can show this by considering two cases one when the eigenfunctions are degenerate and when it is not.

**None Degenerate Case:** Suppose the operator \(\hat{A}\) commutes with operator \(\hat{B}\)

\[ [\hat{A}, \hat{B}] = 0 \] \hspace{1cm} (12)
and has a nondegenerate eigenfunction \( u_a (x) \) with a corresponding eigenvalue \( a \)

\[
\hat{A} u_a (x) = a u_a (x).
\]  \hspace{1cm} (13)

when the operator \( \hat{B} \) acts on this eigenvalue equation

\[
\hat{B} \hat{A} u_a (x) = a \hat{B} u_a (x).
\]  \hspace{1cm} (14)

when the two operators commute, we have

\[
[\hat{A}, \hat{B}] = 0 \Rightarrow \hat{B} \hat{A} = \hat{A} \hat{B}
\]  \hspace{1cm} (15)

so that

\[
\hat{A} \left( \hat{B} u_a (x) \right) = a \left( \hat{B} u_a (x) \right).
\]  \hspace{1cm} (16)

This means that the function

\[
v (x) = \hat{B} u_a (x)
\]

is the eigenfunction of the operator \( \hat{A} \), with eigenvalue \( a \), we must conclude that

\[
\hat{B} u_a (x) = b u_a (x)
\]
in order to satisfy the equation

\[ \hat{A} \left( \hat{B} u_a (x) \right) = a \left( \hat{B} u_a (x) \right) \]

\[ \Rightarrow \hat{A} \left( b u_a (x) \right) = a \left( b u_a (x) \right) \]

\[ \Rightarrow \hat{A} u_a (x) = a u_a (x). \]  \hspace{1cm} (17)

Therefore, we can conclude that two commuting operators \( \hat{A} \) and \( \hat{B} \) have a simultaneous eigenfunction which we denote as \( v_{ab} \) such that

\[ \hat{A} v_{ab} (x) = a v_{ab} (x), \hat{B} v_{ab} (x) = b v_{ab} (x). \]  \hspace{1cm} (18)

*Degenerate Case:* Suppose the operator \( \hat{A} \) commutes with operator \( \hat{B} \)

\[ \left[ \hat{A}, \hat{B} \right] = 0 \]  \hspace{1cm} (19)

and has two degenerate eigenfunctions \( u^{(1)}_a (x) \) and \( u^{(2)}_a (x) \) with the same eigenvalue \( a \),

\[ \hat{A} u^{(1)}_a (x) = a u^{(1)}_a (x) \]  \hspace{1cm} (20)

\[ \hat{A} u^{(2)}_a (x) = a u^{(2)}_a (x). \]  \hspace{1cm} (21)
If we add these two functions we find
\[
\hat{A} \left( u_a^{(1)}(x) + u_a^{(2)}(x) \right) = a \left( u_a^{(1)}(x) + u_a^{(2)}(x) \right)
\]  \hspace{1cm} (22)

and when we subtract
\[
\hat{A} \left( u_a^{(1)}(x) - u_a^{(2)}(x) \right) = a \left( u_a^{(1)}(x) - u_a^{(2)}(x) \right).
\] \hspace{1cm} (23)

This suggests that the linear combination of the two degenerate eigenfunctions are also the eigenfunction for the operator \(A\) with eigenvalue \(a\). Therefore we may define two eigenfunctions which are the linear combination of these two degenerate eigenfunctions as
\[
\begin{align*}
  v_a^{(1)}(x) &= u_a^{(1)}(x) + u_a^{(2)}(x) \\
  v_a^{(2)}(x) &= u_a^{(1)}(x) - u_a^{(2)}(x)
\end{align*}
\]

such that
\[
\hat{A} v_a^{(1)}(x) = av_a^{(1)}(x) \hspace{1cm} (24)
\]
\[ \hat{A}v_a^{(2)}(x) = a v_a^{(2)}(x). \quad (25) \]

If we define two eigenfunctions as a linear superposition of these two functions as

\[ \hat{B}\hat{A}v_a^{(1)}(x) = a\hat{B}v_a^{(1)}(x) \quad (26) \]

\[ \hat{B}\hat{A}v_a^{(2)}(x) = a\hat{B}v_a^{(2)}(x) \quad (27) \]

When the two operators commute, we have

\[ [\hat{A}, \hat{B}] = 0 \Rightarrow \hat{B}\hat{A} = \hat{A}\hat{B} \quad (28) \]

so that

\[ \hat{A}\left(\hat{B}v_a^{(1)}(x)\right) = a\left(\hat{B}v_a^{(1)}(x)\right) \quad (29) \]

\[ \hat{A}\left(\hat{B}v_a^{(2)}(x)\right) = a\left(\hat{B}v_a^{(2)}(x)\right) \quad (30) \]

These two equations mean that the function

\[ v^{(1)}(x) = \hat{B}v_a^{(1)}(x), v^{(2)}(x) = \hat{B}v_a^{(2)}(x) \]

are the eigenfunction of the operator \( \hat{A} \), with eigenvalue \( a \), we must then conclude that

\[ \hat{B}v_a^{(1)}(x) = b_1v_a^{(1)}(x), \hat{B}v_a^{(2)}(x) = b_2v_a^{(2)}(x) \]
in order to satisfy the equation

\[
\hat{A} \left( \hat{B}v_a^{(1)}(x) \right) = a \left( \hat{B}v_a^{(1)}(x) \right)
\]

\[
\Rightarrow \hat{A} \left( b_1v_a^{(1)}(x) \right) = a \left( b_1v_a^{(1)}(x) \right)
\]

\[
\Rightarrow \hat{A}v_a^{(1)}(x) = av_a^{(1)}(x).
\] (31)

\[
\hat{A} \left( \hat{B}v_a^{(2)}(x) \right) = a \left( \hat{B}v_a^{(2)}(x) \right)
\]

\[
\Rightarrow \hat{A} \left( b_2v_a^{(2)}(x) \right) = a \left( b_2v_a^{(2)}(x) \right)
\]

\[
\Rightarrow \hat{A}v_a^{(2)}(x) = av_a^{(2)}(x).
\] (32)

Therefore, we can conclude that two commuting operators \( \hat{A} \) and \( \hat{B} \) have a simultaneous eigenfunction which we denote as \( v_{ab} \) such that

\[
\hat{A}v_{ab}(x) = av_{ab}(x), \hat{B}v_{ab}(x) = bv_{ab}(x). \] (33)

Note that here the second operator \( \hat{B} \) can have different eigenvalues for the two degenerate eigenfunctions

\[
v_{ab}(x) = v_a^{(1)}(x), v_{ab}(x) = v_a^{(2)}(x)
\]
If a given operator $\hat{A}$ has $n$ degenerate eigenfunctions, then there exists another $n-1$ operators that commute with this operator and have the same eigenfunction. One good example is the energy operator ($\hat{H}$), the square of the angular momentum operator ($\hat{L}^2$), and the z component of the angular momentum operator ($\hat{L}_z$) in a hydrogen atom. The energy operator has three degenerate eigenfunctions which are the eigenfunctions of the operators $\hat{L}^2$ and $\hat{L}_z$. Such kind of commuting operators are called completing set of commuting observable and have one important property. These observable can be simultaneously measured with zero uncertainty. This is basically a consequence of the uncertainty relation

$$\left(\Delta \hat{A}\right)^2 \left(\Delta \hat{B}\right)^2 \geq \frac{1}{4} \left\langle i \left[\hat{A}, \hat{B}\right]\right\rangle^2.$$  

(34)