A Brief Review of Theory of Interest

Don Hong
Department of Mathematical Science
Middle Tennessee State University
Murfreesboro, TN 37132, dhong@mtsu.edu
Chapter 1

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1.1 Introduction

Actuaries determine values now that measure the financial impact of events that may occur at some unknown future time. They determine these values using business judgment, probability theory, and interest theory. You need to look elsewhere to learn business judgment. The presentation here is brief – I assume you’re already familiar with the material or can learn it easily. For treatments both broader and more detailed, see [Kellison 1991].

1.2 Time value of money

Which would you rather have – $1000 now or $1000 one year from now? How about $1030 a year from now? Or $1060? To determine a value now that measures the financial impact of a future event, the actuary needs the concept of the present value of future money. You probably view $1000 now as having more value than has $1000 a year from now. This may partly be due
to inflation, since $1000 in the future may not buy as much as it could now. But it’s also because you could invest today’s $1000 and see it grows to a larger amount in a year.

**Example 1.1.** Suppose that $1000 invested today would grow to $1045 a year from now. In this circumstance, you might view $1045 one year from now and $1000 now as equivalent – ignoring issues like you possibly needing money more today. Similarly, you’d view $1045/1000 = $1.045 one year from now as equivalent to $1 now, and $1 a year from now as equivalent to $1000/1045 = $0.9569 today. That is, you’d view $1.45 as the future value one year from now of $1 today, and .9569 as the present value today of $1 a year from now.

The discussion in Example 1.1 demonstrates the time value of money. Actuaries needs general mathematical tools to discuss this issue and specifically to measure the present value now equivalent to $1 at a particular time in the future as well as the future value of $1 now. The general mathematical tools for treating the time value of money are built on the concept of an accumulation function.

**Definition 1.2.** An Accumulation function is a real-value function $a$ defined for $t \geq 0$, satisfying $a(0) = 1$, and such that $1$ invested at time 0 grows to $a(t)$ at future time $t$.

Here, we use $t = 3$ to denote 3 time periods. You probably assume it to stand for 3 years. You should recognize, however, that on occasion it is more convenient to measure time in some other unit such as months, days, weeks, half-years, quarter-years, or $\frac{1}{m}$-years. More generally, you can think of time as measured in general unit of a period – abbreviated PRD.

Just as in Example 1.1 we viewed $1.045$ as the future value at time-1 of a dollar now, we more generally view $a(t)$ as the time-$t$ future value of a dollar now – taking ‘now’ as time-0. Again as in that example, we’ll also view $1$ as the $1/a(t)$ now; that is, the time-0 present value of a time-$t$ dollar is given by $1/a(t)$. This important expression – the discount function – is
denoted by $v(t)$:

$$v(t) = \frac{1}{a(t)}. \quad (1.1)$$

These two functions – the accumulation function and the discount function – are the key tools for analyzing the time value of money. Remember: $a(t)$ is the time-$t$ value of a time-0 dollar, while $v(t)$ is the time-0 value of a time-$t$ dollar.

Now for a little more complicated question: What is the time-20 value of a time-15 $100? We’ve agreed to view that time-15 $100 as equivalent to $100v(15)$ at time-0, and each time-0 dollar as equivalent to $a(20)$ time-20 dollars. So this makes the 100 time-15 dollars equivalent to $100v(15)a(20)$ time-20 dollars. Similarly, it seems reasonable to view 1 time-$t$ dollar as equivalent to $a(s)v(t)$ time-$s$ dollars. Let’s agree to this as a definition.

**Definition 1.3.** The time-$t$ value $X_t$ and the time-$s$ value $X_s$ of the same quantity of money are defined by the equivalent formula

$$X_s = a(s)v(t)X_t$$

$$X_t = a(t)v(s)X_s.$$  

**Example 1.4.** Suppose that the accumulation function is given by $a(t) = 1 + 0.05t$. What is the time-15 value of 100 time-10 dollars?

According to Definition 1.3, it is

$$\$100v(10)a(15) = \$100 \frac{a(15)}{a(10)} = \$100 \frac{1.75}{1.5} = \$116.67.$$  

How about the time-8 value of 100 time-3 dollars?

Definition 1.3 gives

$$\$100v(3)a(8) = \$100 \frac{a(8)}{a(3)} = \$100 \frac{1.4}{1.15} = \$121.74.$$  

Note the differences in the two answers, despite $\$100$ having grown for five years in both cases. The message here is that
in general, the accumulation or discount of money over time depends not only on the length of the time interval but also on where in time the interval lies.

Problems

1. If you invest $1000 at time-0 and \( a(t) = 1 + 0.06t \), how much will you have at time-6? How much would you need to invest at time-0 in order for it to have grown to $2000 at time-10?

2. Suppose that \( a(t) = 1 + 0.5t + 0.001t^2 \). Find the time-0 value of $500 at time-8.

3. Suppose that time is measured in half-years and that \( a(t) = (1.04)^t \). If you invest $2000 at \( t = 0 \), to how much will it have grown when \( t = 3 \)? To how much after three years?

4. Suppose that \( t \) measures months and that \( v(t) = e^{-0.05t} \). Find the time-0 present value of $800 to be given you two years from now.

5. Find the time-7 value of 800 time-20 dollars, if \( a(t) = 1 + 0.03t \).

6. Suppose that \( a(t) = 1 + 0.07t \) and that $\text{P}$ at time-0 together with $\text{P}$ at time-1 is equivalent to $2000 at time-2. Find $\text{P}$.

1.3 Simple and Compound Interest

What do accumulation and discount functions look like in practice? Consider for example simple interest. $100 deposited in a saving account that earns 5% yearly simple interest would grow to $105 in one year, $110 in two years, $115 in three years, and more generally $100 + 5n$ dollars in an integer $n$ years. This means
that \(a(n) = 1 + .05n\) for integer \(n\). If we assume that the investment can be withdrawn at any future time \(t\) – not just after an integer number of years – with the same formula that applied for integers, then \(a(t) = 1 + .05t\) There is nothing special about the rate 5% or the period of a year, of course; assuming that money can be withdrawn as just stated provides the general definition of simple interest’s accumulation and discount functions.

**Definition 1.5.** If time \(t\) is measured in PRDs and money invested at time-0 earns simple interest at the PRDly rate \(r\), then the accumulation function is defined by \(a(t) = 1 + rt\) and the discount function by \(v(t) = 1/(1 + rt)\).

**Example 1.6.** Consider 6% yearly simple interest. Then \(a(t) = 1 + .06t\), and $1000 at time-0 accumulates to $1000(1 + .06 \cdot 3) = $1180 at time-3, which is three years later. Consider instead 3% half-yearly simple interest. Then this time \(a(t) = 1 + .03t\), and $1000 at time-0 accumulates to $1000(1 + .03 \cdot 3) = $1090 at time-3, which is three half-years later. What would this second fund equal after three years? Three years equals six half-years and so corresponds to \(t = 6\) in the second investment; the fund would accumulate to $1000(1 + .03 \cdot 6) = $1180 at time-6. Note that both funds total $1180 after three years, which is \(t = 3\) for the first fund and \(t = 6\) for the second. This hints that in general simple interest at the yearly rate \(r\) is the same as simple interest at the half-yearly rate \(r/2\). Problem 4 in this section says the more general result holds for \(\frac{1}{m}\)-yearly simple interest rates.

Simple interest is rarely used in the actuarial applications addressed in this book. More common is compound interest. $100 deposited in a savings account that earns 5% yearly compound interest would grow to $100(1.05) = $105 in one year, $100(1.05)^2 = $110.25 in two years, $100(1.05)^3 = 115.7625$ in three years, and more generally $100(1.05)^n$ in an integer \(n\) years. This means that \(a(n) = (1.05)^n\) for integer \(n\). If we assume that the investment can be withdrawn at any future time \(t\) – not just after an integer number of years – with the same formula that
applied for integers, then \( a(t) = (1.05)^t \). Again there is nothing special about the rate 5% or the period of a year, of course; assuming that money can be withdrawn as just stated provides the general definition of compound interest’s accumulation and discount functions.

**Definition 1.7** If time \( t \) is measured in PRDs and money invested at time-0 earns *compound interest* at the PRDly rate \( r \), then the accumulation function is defined by \( a(t) = (1+r)^t \) and the discount function by \( v(t) = (1+r)^{-t} \).

**Example 1.8.** Consider 6% yearly compound interest. Then by Definition 1.7, \( a(t) = (1+0.06)^t \), and $1000 at time-0 accumulates to \( $1000(1+0.06)^3 = $1191.016 \) at time-3, which is three years later. Consider instead 3% half-yearly compound interest. Then this time \( a(t) = (1+0.03)^t \), and $1000 at time-0 accumulates to \( $1000(1+0.03)^3 = $1092.727 \) at time-3, which is three half-years later. What would this second fund equal after three years? Three years equals six half-years and so corresponds to \( t = 6 \) in the second investment; the fund would accumulate to \( $1000(1+0.03)^6 = $1194.0523 \) at time-6. Unlike the simple interest case of Example 1.6, the two funds differ after three years. This shows that 6% yearly compound interest and 3% half-yearly compound interest are different. But notice that \( (1.03)^6 = [(1.03)^2]^3 = (1.0609)^3 \). A time-0 investment of $1000 at 6.09% yearly compound interest would accumulate in three years to \( $1000(1.0609)^3 = $1194.0523 \), exactly the same as after three years at 3% half-yearly compound interest. This indicates that 3% half-yearly compound interest and 6.09% yearly compound interest are identical.

There is nothing special about 6% or half years in Example 1.8. The general result is important enough to be dignified as a theorem.

**Theorem 1.9.** Suppose that one dollar is invested at time-0 at an \( \frac{1}{m} \)-yearly compound interest rate of \( r \). Then this is equivalent to invest one dollar at time-0 at yearly compound interest rate of \( (1+r)^m - 1 \).
Proof. Let \( y \) denote any number of years, integer or otherwise. For the first investment, \( y \) years corresponds to \( t = my \) basic periods, so that the invested dollar will have grown to \((1 + r)^{my}\) dollars after \( y \) years. Let \( \tilde{r} \) denote the rate \( \tilde{r} = (1 + r)^{m} - 1 \) for the second investment. For that investment, \( y \) years corresponds to \( t = y \) basic periods, so that the invested dollar will have grown to

\[
(1 + \tilde{r})^y = [(1 + r)^{m}]^y = (1 + r)^{my}
\]
dollars after \( y \) years, the same total as for the first investment. Thus, the two investments are equivalent.  

Look back at Definition 1.3 and Example 1.4 on the time value of money, where you saw that in general the growth of an investment from time-\( t \) to time-\( s \) depends on \( t \) and \( s \) separately – not just on the length \( s - t \) of the time investment. This is not true for compound interest, as the following simple but important theorem shows.

**Theorem 1.10.** Let \( a(t) = (1 + r)^t \) be the accumulation function for compound interest at the PRDly rate \( r \). Then the time-\( t \) value \( X_t \) and time-\( s \) value \( X_s \) of the same quantity of money are defined by the equivalent formulas

\[
X_s = (1 + r)^{s-t}X_t
\]

\[
X_t = (1 + r)^{t-s}X_s.
\]

That is, the time value of money under compound interest depends only on the length \( s - t \) of time elapsed between the two evaluation times.

**Proof.** Just substitute \( a(t) = (1 + r)^t \) and \( v(t) = (1 + r)^{-t} \) into the formulas given in Definition 1.3.  

What does Theorem 1.10 say about compound interest at the PRDly rate \( r \)? For one thing, if you have \( X \) dollars at time-\( t \) then their future value \( h \) periods later equals \( X(1 + r)^h \) dollars regardless of the value of \( t \). For another, those \( X \) time-\( t \) dollars were worth \( X(1 + r)^{-h} \) dollars \( h \) years earlier, again regardless of
the value of $t$. So, with compound interest, the time-0 reference point is irrelevant – you need know only the elapsed time $h$ to discount future values back in time by the factor $(1 + r)^{-h}$ or to accumulate past values forward in time by the factor $(1 + r)^h$.

*This is one of the main tools actuaries use to measure the present value of the impact future financial events.*

**Problems**

1. Find the yearly simple interest rate so that $1000$ invested at time-0 will grow to $1300$ in four years.

2. The monthly simple interest rate is $5\%$. How much need be invested at time-0 so as to grow to $2000$ in two years?

3. The yearly simple interest rate is $10\%$. Find the time-3 value of $1000$ time-0 dollars together with $500$ time-8 dollars.

4. Show that the result of Example 1.6 is true more generally – that simple interest at the $\frac{1}{m}$-yearly rate $r$ is the same as simple interest at the yearly rate $rm$.

5. Find the yearly compound interest rate so that $1000$ invested at time-0 will grow to $1300$ in four years.

6. The monthly compound interest arte is $5\%$. How much need be invested at time-0 so as to grow to $2000$ in two years.

7. The yearly compound interest rate is $10\%$. Find the time-3 value of $1000$ time-0 dollars together with $500$ time-8 dollars.

8. The half-yearly compound interest rate is $3\%$. Suppose that $8000$ time-1 dollars together with $6000$ time-5 dollars are equivalent to $14000$ $y$ years after time-0. Find $y$. 
9. Find the yearly compound interest rate equivalent to .5% monthly compound interest. Find the equivalent two-yearly compound rate.

10. Suppose that .03% daily compound interest is equivalent to 4% PRDly compound interest. How long is the period PRD?

11. Time is measured in years and you want $1200 at time-0 plus $600 at time-2 plus $300 at time-4 to accumulate to $2500 at time-5. What yearly compound interest rate would accomplish this? What monthly compound rate?

12. $1000 now plus $2000 one year from now will accumulate to how much three years from now at 6% yearly compound interest?

13. $X now plus $500 one year from now will accumulate to $1000 after two additional years at 4.5% yearly compound interest. Find $X$.

14. How many years need pass at 3% half-yearly compound interest for $500 to grow to $1000?

15. Find the monthly compound interest rate at which $1000 to be paid two years from now has a present value today of $800.

16. Payment of $P one year from now and $P two years from now exactly repay a present debt of $5000 at 6% yearly compound interest. Find $P$.

17. Find the quarterly – that is, quarter-yearly – compound interest rate so that $500 half a year from now and $500 one year from now fairly repays a debt that is presently $950.
18. A $1600 present debt is to be repaid by payments of $500 in $y$ years and $1500 after an additional $2y$ years at 7% yearly compound interest. Find $y$.

### 1.4 Effective rates for a period

Actuaries use the concept of effective rate of various types to compare the behavior of different investments.

Perhaps the most common such rates is the effective rate of interest for a period, which is just the compound interest rate for that period that would have produced the same growth during that period. If the investment is described as usual by its accumulation function, then the effective interest rate for the period from, say 3.2 to $3.2 + h$ is simply given by

\[
\frac{a(3.2 + h) - a(3.2)}{a(3.2)}
\]

as an $h$-ly rate – if $h$ equals a week, then the above is a weekly compound interest rate. But comparing investments by using effective interest rates over periods of different lengths is about as convenient as comparing the weights of apples and oranges by using ounces for one and kilograms for the other. For convenience, effective rates are usually computed over a period of unit length in whatever units are being used to measure $t$ – our PRD, typically a year – so that usually $h = 1$.

Sometimes financial transactions are described in terms of discount rates rather than interest rates. For example if you ask to borrow $1000 for one year and are told that you will be charged a 5% discount rate, then you will actually be handed $(1000)(1 - .05)$ by the lender and will have to repay $1000 in one year. From the interest-rate viewpoint, you will have paid $50$ interest on a $950 loan for an effective yearly interest rate of \(\frac{1000 - 950}{950} = .05623\), or 5.623%. Note that this interest rate
could also be computed directly from the discount rate as \( \frac{0.05}{1-0.05} \). That is, an interest rate \( i \) and a discount rate \( d \) describe the same situation when \( i = \frac{d}{1-d} \), which can be solved for \( d \) to give \( d = \frac{i}{1+i} \).

**Definition 1.11.** Suppose that time \( t \) is measured in PRDs and that \( a \) is the accumulation function that describe the time value of money. Then the effective PRDly rate of interest \( i_t \) and the effective PRDly rate of discount \( d_t \) for the interval \( t-1 \) to \( t \) are given by

\[
i_t = \frac{a(t) - a(t-1)}{a(t-1)} \quad (1.2)
\]

\[
d_t = \frac{a(t) - a(t-1)}{a(t)} \quad (1.3)
\]

Therefore, we have \( t_t = \frac{d_t}{1-d_t} \), and \( d_t = \frac{i_t}{1+i_t} \).

Note that \( i_t \) and \( d_t \) refer to the interval from \( t-1 \) to \( t \) so that \( i_1 \) is the effective rate during the first period after \( t = 0 \), \( i_2 \) during the second period, and so on.

**Example 1.12.** Consider yearly simple interest of 5%, so that \( a(t) = 1 + 0.05t \). Then \( i_t = \frac{0.05}{(0.95 + 0.05t)} \), which shows that \( i_t \) can indeed depend on and change with \( t \) – for instance, \( i_2 = 0.0476 \) while \( i_{11} = 0.0333 \).

Unlike in Example 1.12, effective rates do not vary with \( t \) in the case of compound interest. Suppose for example that the compound interest rate is \( r \), so that \( a(t) = (1+r)^t \). Then

\[
i_t = \frac{(1+r)^t - (1+r)^{t-1}}{(1+r)^{t-1}} = r
\]

regardless of the value of \( t \). This also of course gives \( d_t = \frac{r}{(1+r)} \), again independent of \( t \). These constant effective rates are traditionally denoted by \( i \) and \( d \), respectively. Note also that the effective interest rate \( i \) is identical with the compound interest rate \( r \); this essentially allows replacing the phrase ‘the compound interest rate \( i \)’ by ‘the effective interest rate \( i \)’ or even by ‘the effective discount rate \( d \)’ if \( d = \frac{r}{(1+r)} \).
Since this book – and actual practice – typically assumes compound interest, I’ll summarize as a theorem the important facts proved here and earlier about that case; the meaning of the symbol δ (called the force of interest in the theorem) should become clear in the next section.

**Theorem 1.13.** Consider compound interest at the PRDly rate $i$, with time $t$ measured in PRDs so that the accumulation function is given by $a(t) = (1 + i)^t$. Let $d = \frac{i}{(1+i)}$, and $δ = \ln(1 + i)$. Then

\[
v = \frac{1}{1+i} = 1 - d = e^{-δ}, \quad i = \frac{d}{1-d}, \quad δ = \ln(1 + i),
\]

the effective PRDly interest rate is constant at $i$, and the effectively PRDly discount rate is constant at $d$. Moreover, the accumulation function and the discount function can be given in terms of $δ$ (the force of interest) and $v$ (the discount factor) by $a(t) = e^{δt}$ and $v(t) = v^t = e^{-δt}$.

**Problems**

1. Suppose $a(t) = 1 + .06t$. Find the effective rate of interest for the period $t$ to $t + .5$; for $t$ to $t + 1$; and for $t$ to $t + 2$.

2. Suppose $a(t) = (1+.06)^t$. Find the effective rate of interest for the period $t$ to $t + .5$; for $t$ to $t + 1$; and for $t$ to $t + 2$.

3. The accumulation function $a(t) = \frac{1}{(1-.06t)}$ is said to describe simple discount at the rate of 6%. Find the effective rate of interest at $t = 3, 11$, and general $t$.

4. The accumulation function $a(t) = \frac{1}{(1-r)^t}$ is said to describe compound discount at the rate $r$. Find the effective rate of interest $i_t$ and the effective rate of discount $d_t$ for $t = 3, 11$, and general $t$. 
5. Show that compound discount at the rate \( r \) as defined in Problem 4 is equivalent to compound interest at the rate \( \frac{r}{1-r} \).

6. Find the present value of $1000 five years from now if the yearly effective interest rate is 6%.

7. At what half-yearly effective interest rate will $500 grow to $700 in three years?

8. In how many years will $1000 grow to $2000 at the effective monthly interest rate of .5%?

9. A debt of $2000 is to be repaid with payments of $P$ one year from now and $2P$ two years from now. Find $P$ so as to fairly repay the debt at a yearly effective discount rate of 5%.

10. Find the numerical values of the constants \( i, d, v \), and \( \delta \) from Theorem 1.13 if \( a(t) = (.95)^{-t} \).

11. Show that the constants of Theorem 1.13 satisfy:

   (a) \( i = (1 + i)d \)
   (b) \( d = iv \)
   (c) \( i = v^{-1} - 1 \)
   (d) \( 1 + i = \frac{1}{1-d} \)
   (e) \( i - d = id \)
   (f) \( 1 + i = e^\delta \)
   (g) \( v = e^{-\delta} \).

12. Use calculus and power series to show that the constants in Theorem 1.13 satisfy:

   (a) \( v = 1 - d = \sum_{j=0}^{\infty}(-1)^j i^j \)
(b) \( v = \sum_{j=0}^{\infty} \frac{(-1)^j j^j}{j!} \)

(c) \( \delta = \sum_{j=1}^{\infty} (-1)^{j+1} j^j \)

(d) \( d = \sum_{j=1}^{\infty} \frac{(-1)^j j^j}{j!} \)

(e) \( i = \sum_{j=1}^{\infty} d^j \)

(f) \( i = \sum_{j=1}^{\infty} \frac{d^j}{j!} \)

(g) \( \delta = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} i^j}{j} \)

(h) \( \delta = \sum_{j=1}^{\infty} \frac{d^j}{j} \)

## 1.5 Nominal rates and the force of interest

Many financial institutions describe interest in terms of nominal yearly interest rates – rates that are yearly in name only. Thus you’ll hear of 8% nominal yearly interest compounded quarterly, or 6% nominal yearly interest convertible monthly. Actually, the ‘yearly’ is usually understood rather than stated: 18% nominal interest payable monthly. And ‘compounded’, ‘convertible’, and ‘payable’ are used interchangeably.

What do these terms mean? The nominal yearly interest rate of 8% convertible quarterly actually is identical with compound interest at the quarterly rate of \( \frac{8\%}{4} = 2\% \). Similarly nominal interest of 8% compounded monthly is identical with compound interest at the monthly rate of \( \frac{8\%}{12} = \frac{2\%}{3} \). In general, nominal interest at the rate \( j \) payable \( \frac{1}{m} \)-ly is identical with compound interest at the \( \frac{1}{m} \)-yearly rate \( \frac{j}{m} \).

The \( \frac{1}{m} \)-yearly compound interest rate \( j/m \) of course has a corresponding \( \frac{1}{m} \)-yearly compound discount rate

\[
\frac{j/m}{1 + j/m};
\]
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$m$ times this true rate is again a nominal rate, this time the nominal discount rate convertible $\frac{1}{m}$ly. By using Theorem 1.9 with $r = \frac{j}{m}$, you can easily convert the $\frac{1}{m}$ly effective interest rate $\frac{j}{m}$ into the true yearly effective rate $i = (1 + \frac{j}{m})^m - 1$.

In general practice, a nominal interest rate convertible $\frac{1}{m}$ly is denoted by the symbol $i^{(m)}$, and the corresponding nominal discount rate by $d^{(m)}$, so that $i$ and $d$ can be reserved for the corresponding true yearly rates.

**Definition 1.15.** Nominal (yearly) interest convertible (or compounded, or payable) $\frac{1}{m}$ly at the nominal rate $i^{(m)}$ is defined to mean compound interest at the $\frac{1}{m}$-yearly rate $i^{(m)}_m$. The corresponding nominal (yearly) discount convertible (or compounded, or payable) $\frac{1}{m}$ly at the nominal rate $d^{(m)}$ is defined to mean nominal interest convertible $\frac{1}{m}$ly at the nominal rate $i^{(m)} = -\frac{d^{(m)}}{1-(d^{(m)}/m)}$.

The above definition of $i^{(m)}$ in terms of $\frac{1}{m}$-yearly effective rate $i^{(m)}/m$ as

$$\frac{i^{(m)}}{m} = \frac{(d^{(m)}/m)}{1-(d^{(m)}/m)}.$$ 

As noted above, the yearly effective interest rate $i$ is therefore $(i + \frac{i^{(m)}}{m})^m - 1$, which gives $i^{(m)} = m[(1 + i)^{\frac{1}{m}} - 1]$; similarly, the yearly effective discount rate $d$ and $d^{(m)}$ can be expressed in terms of each other. And finally, the $\frac{1}{m}$ly discount factor is given by $\frac{1}{(1+i^{(m)}/m)} = (\frac{1}{1+i})^{\frac{1}{m}} = v^{\frac{1}{m}}$. This proves the following.

**Theorem 1.15.** Nominal interest rate $i^{(m)}$ and nominal discount rate $d^{(m)}$ convertible $\frac{1}{m}$ly, their equivalent effective yearly interest and discount rates $i$ and $d$, and the discount factor $v$, are related by the following formulas:

$$i^{(m)} = \frac{d^{(m)}}{1-d^{(m)}/m} = m[(1 + i)^{\frac{1}{m}} - 1],$$

$$d^{(m)} = \frac{i^{(m)}}{1+i^{(m)}/m} = m[1 - (1 - d)^{\frac{1}{m}}],$$
\[
\frac{1}{v^m} = \frac{1}{1 + i^{(m)}/m} = 1 - d^{(m)}/m,
\]

\[
i = (1 + i^{(m)}/m)^m - 1, \quad d = 1 - (1 - d^{(m)}/m)^m.
\]

**Example 1.16.** Suppose that \( i = 5\% \) – that is, money earns 5% yearly compound interest. This of course makes \( d = 0.05/1.05 = 0.04762 \), or 4.762%. What is the equivalent monthly compound interest rate? Just \( i^{(12)}/12 \) as computed from

\[
i^{(12)}/12 = (1.05)^{\frac{1}{12}} - 1 = .004074.
\]

This gives \( i^{(12)} = .04889 \). The corresponding \( d^{(12)} \) equals .04869. Note here that \( d < d^{(12)} < i^{(12)} < i \), a result which generalizes.

**Example 1.17.** Consider the rate of 3% per annum convertible every two years. The \( i^{(1/2)} = 3\% \). The two-yearly rate \( j = i^{(1/2)} = 6\% \). The yearly effective interest rate \( i = (1+j)^{1/2} - 1 = .

You should be able to solve Problems 1 to 8 in this section now.

In the following, we discuss the force of interest. Suppose that you want to measure how well an investment is doing around a particular point in time, say \( t \), where as usual a one dollar time-0 investment is worth \( a(t) \) time-\( t \) dollars and \( t \) is measured in years. Then

\[
i_t = [a(t + 1) - a(t)]/a(t)
\]

gives the yearly effective rate for the period \( t \) to \( t + 1 \). If the investment varies a lot around time \( t \), however, this year-long measure might not reflect what’s really happening near \( t \). How about \( i^{(2)} = \frac{a(t+1/2)-a(t)}{a(t)}? \) This is the half-yearly effective rate over \( t \) to \( t + \frac{1}{2} \), which could also be described by the equivalent nominal rate convertible semi-annually \( i^{(2)} \). But if there is a lot of financial action around \( t \), this half-year measure still might not reflect what’s happening near \( t \). Imagine then taking a large positive integer \( m \) and computing the yearly nominal interest rate \( i_t^{(m)} \) convertible \( \frac{1}{m} \)-ly equivalent to the effective \( \frac{1}{m} \)-yearly rate from \( t \) to

\[
i_t^{(m)} = m \frac{a(t + \frac{1}{m}) - a(t)}{a(t)} = \frac{a(t + \frac{1}{m}) - a(t)}{\frac{1}{m}} \frac{1}{a(t)}.
\]
As $m$ gets bigger, this nominal yearly rate convertible $\frac{1}{m}$ly represents better and better what is happening near the time $t$. But it’s also clear from the calculus definition of the derivative that, as $m$ tends to infinity – the expression above tends to $a'(t)/a(t)$. Thus $a'(t)/a(t)$ can be interpreted as the nominal yearly interest rate convertible continuously that describes the performance of the investment at the instant $t$.

**Definition 1.18.** Suppose that the time value of money is described by the accumulation function $a$. Then the force of interest $\delta_t$ at time $t$ is defined by $\delta_t = \frac{a'(t)}{a(t)}$.

Note that the accumulation function $a$ can be reconstructed from the force of interest:

$$a(t) = e^{\int_0^t \delta_r \, dr}.$$

(1.4)

**Example 1.19.** Consider simple interest at the yearly rate $r$, so that $a(t) = 1 + rt$. Then $\delta_t = \frac{a'(t)}{a(t)} = \frac{r}{1+rt}$ clearly varies with $t$.

But $\delta_t$ is a constant independent of $t$ in the context of compound interest; in fact, $\delta_t$ equals the constant $\delta$ introduced mysteriously in Theorem 1.13. This is important.

**Theorem 1.20.** Consider compound interest at the PRDly rate $i$, with time $t$ measured in PRDs so that the accumulation function is given by $a(t) = (1 + i)^t$. Then the force of interest $\delta_t$ is constant and equals the number $\delta = \ln(1 + i)$ introduced in Theorem 1.13.

**Proof.** Recall from Theorem 1.13 that $a(t) = e^{\delta t}$. Therefore

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{\delta e^{\delta t}}{e^{\delta t}} = \delta$$

as claimed. □

Look back at the intuitive derivation of the force of interest in light of Theorem 1.20 on the compound-interest case. $\delta_t$ was defined as the limit as $m$ tends to infinity of nominal yearly interest rates convertible $\frac{1}{m}$ly; but under compound interest those
rates equal $i^{(m)}$. This means that $i^{(m)}$ tends to $\delta$. Moreover, the formula in Theorem 1.16 for $d^{(m)}$ in terms of $i^{(m)}$ shows that also $d^{(m)}$ tends to $\delta$. For more information on the relationships among $d^{(m)}$, $i^{(m)}$, and $\delta$, see problems 14, 15 and 16.

**Problems**

1. (a) How much need you invest for five years so that it will grow to $5000 if it earns 6% nominal interest convertible semi-annually? (b) If payable quarterly? (c) If compounded monthly?

2. (a) How much need you invest for five years so that it will grow to semi-annually? (b) If payable quarterly? (c) If compounded monthly?

3. (a) How many years does it take an investment to double if it earns 5% nominal interest convertible semi-annually? (b) If payable weekly? (c) If compounded daily?

4. If time $t$ is measured in years, find formulas for $a(t)$ under:
   (a) $i^{(m)}$ nominal interest convertible $\frac{1}{m}$ly; (b) $d^{(m)}$ nominal discount convertible $\frac{1}{m}$ly.

5. Given that $i = .06$, find: (a) $i^{(2)}$, (b) $d^{(4)}$, (c) $i^{(12)}$, (d) $d^{(12)}$.

6. Given that $d = .06$, find: (a) $i^{(2)}$, (b) $d^{(4)}$, (c) $i^{(12)}$, (d) $d^{(12)}$.

7. Given that $i^{(2)} = .06$, find: (a) $i$, (b) $d$, (c) $i^{(12)}$, (d) $d^{(6)}$.

8. Suppose that $a(t) = .05t + .001t^2$ with $t$ in years. For $m = 4, 12,$ and $52$, calculate the nominal yearly interest rate $i^{(m)}_2$ convertible $\frac{1}{m}$ly measuring the investment’s performance from $t = 2$ to $t = 2 + \frac{1}{m}$. Also compare $\delta_2$ and observe the convergence of $i^{(m)}$ to $\delta_2$. 
9. Given \( a(t) = \exp(0.05t + 0.001t^2) \), find the force of interest. Does this describe compound interest? why or why not?

10. Given \( a(t) = (1.03)^{2t} \), find the force of interest. Does this describe compound interest? why or why not?

11. Given \( \delta_t = 0.06 \), find the force of interest. Does this describe compound interest? why or why not?

12. For each following \( \delta_t \), find the corresponding \( a(t) \): (a) \( \delta_t = 0.05 \frac{1}{1+0.05t} \), (b) \( \delta_t = 0.05 \frac{k}{1+k+0.05t} \).

13. For each following \( \delta_t \), find the corresponding \( a(t) \): (a) \( \delta_t = \frac{r}{1+rt} \), (b) \( \delta_t = \frac{kr}{\sqrt{1+rt}} \).

14. By differentiating the formulas for \( i(m) \) and \( d(m) \) with respect to \( m \), show that \( i(m) \) decreases to \( \delta \) and \( d(m) \) increases to \( \delta \) as \( m \) increases. Goon to conclude that \( d < d(m) < \delta < i(k) < i \) so long as \( \delta > 0, m > 1, \) and \( k > 1 \).

15. Use the fact that \( 1+i(m)/m = e^{\delta/m} \) to show that \( \delta/m \) is the constant force of interest that corresponds to the \( \frac{1}{m} \) yearly compound interest rate \( i(m)/m \) and the corresponding discount rate \( d(m)/m \). Replacing \( i \) by \( i(m) \), \( d \) by \( d(m) \), and \( \delta \) by \( \delta/m \) in Problem 12 in section 1.3, show that the following equations hold:
   (a) \( i(m) = \sum_{j=1}^{\infty} \delta^j \frac{\frac{1}{m}^{j-1}}{j!} \)
   (b) \( d(m) = \sum_{j=1}^{\infty} (-1)^{j+1} \delta^j \frac{\frac{1}{m}^{j-1}}{j!} \).

16. Show that \( d(-m) = i(m) \).

### 1.6 Level annuities-certain

An *annuity-certain* is a regularly-spaced sequence of payments certain to be made for a given period of time. Such payments
arise naturally in many applications – regular payments to repay a loan such as a mortgage, regular deposits to a savings account, and the like. Applications related to the main ideas of this book include regular premium payments on insurance policies, pension payments, and the like.

For example, consider a sequence of three $1 payments spaced a year apart – say at time-0, time-1, and time-2. Use of the discount function \( v(t) \) provides the total value of the annuity payments at time-0: \( 1 + v(1) + v(2) \) dollars. Although it’s certainly possible to discuss annuities in the general context of arbitrary accumulation and discount functions, from now on let’s assume PRDly compound interest at the PRDly interest rate \( i \) so that \( a(t) = (1 + i)^t \) and \( v(t) = v^t \) with \( v = 1/(1 + i) \) as usual. Then the total value of the three payments at the moment of the first is just

\[
1 + v + v^2 = \frac{1 - v^3}{1 - v} = \frac{1 - v^3}{d}.
\]

More generally, consider a sequence of \( n \) payments of $1 spaced exactly one basic period (PRD) apart. With PRDly compound interest at the rate \( i \), the total value of the payments at the moment of the first payment is clearly

\[
1 + v + v^2 + \cdots + v^{n-1} = \frac{1 - v^n}{1 - v} = \frac{1 - v^n}{d}. \tag{1.5}
\]

Values at any other point in time can easily be computed from the above by multiplying by the appropriate discount term \( v^t \) or growth term \( (1 + i)^t \). For example, the value \( n \) periods after the first payments equals

\[
(1 + i)^n \frac{1 - v^n}{d} = \frac{(1 + i)^n - 1}{d}; \tag{1.6}
\]

that 3 periods before the first payments equals

\[
v^3 \frac{1 - v^n}{d};\]
and so on. Standard symbols have been developed to denote the total value of the annuity-certain at special points in time: the moment of the first payment, one period before the first payment, the moment of the last payment, and one period after the last payment.

**Definition 1.21.** For a series of \( n \) payments of 1 are made exactly one basic period (PRD) apart, with compound interest at the PRDly rate \( i \), the symbol \( \ddot{a}_{\overline{n|}} \) denotes the current value of the \( n \) payments at the moment of the first payment; the symbol \( a_{\overline{n|}} \) denotes the current value of the \( n \) payments at the moment of one period before the first payment; the symbol \( s_{\overline{n|}} \) at one period after the last payment; and the symbol \( s_{\overline{n|}} \) at the moment of the last payment. If it is not clear from the context what rate \( i \) is being used, then \( i \) is added as a subscript as in \( \ddot{a}_{\overline{n|i}} \).

This definition emphasizes that the four annuity-certain symbols defined there evaluate the same set of payments –just at different points in time. An alternative approach views them as describing different kinds of annuities: annuities-certain-due and annuities-certain-immediate. An \( n \) year *annuity-certain-due* is a sequence of \( n \) payments that are made at the beginning of each year. Thus, \( \ddot{a}_{\overline{n|}} \) and \( \ddot{s}_{\overline{n|}} \) give the values of this annuity-certain-due at the start and at the finish, respectively, of the \( n \)-year term. On the other hand, an \( n \)-year *annuity-certain-immediate* that starts now has its payments at the end of each year, so that the first one is in one year at time-1 and the last one is in \( n \) years at time-\( n \), which is the end of the \( n^{th} \) year; so, \( a_{\overline{n|}} \) and \( s_{\overline{n|}} \) give the values of this annuity-certain-immediate at the start and at the finish, respectively, of the \( n \)-year term. Equation (1.5) gives the present value of the annuity-certain-due

\[
\ddot{a}_{\overline{n|}} = \frac{1 - v^n}{d},
\]

while equation (1.6) gives the accumulated value of the annuity-certain-due

\[
\ddot{s}_{\overline{n|}} = (1 + i)^n - 1.
\]
Note that each of these symbols has a $d$ in its denominator, while the following annuity-certain-immediate formulas differs only in that an $i$ replace the $d$:

$$a_{\overline{n}|} = v a_{\overline{n}|} = \frac{1 - v^n}{i}, \quad (1.9)$$

and

$$s_{\overline{n}|} = v s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}. \quad (1.10)$$

**Example 1.22.** Suppose that Schmidt borrows $10,000 today at 6% yearly compound interest for five years. One way to repay it would be through a series of five regular payments at the end of each year for five years – that’s a 5-year annuity-certain-immediate. How large would each payment be? If the payment is $P$, then the value now of the five payments at 6% would have to equal the debt:

$$10,000 = P \cdot a_{\overline{5}|.06}.$$ 

Since $a_{\overline{5}|.06} = \frac{1 - v^5}{i} = \frac{1 - (1.06)^{-5}}{.06} = 4.212$, $P = 2373.96$. Suppose, however, that Schmidt can earn 7% compound interest in a special savings account and consider depositing $Q$ there at the start of each year for five years – that is a 5-year annuity-certain-due; how big need $Q$ to be so that it grows to pay off the debt at the end of five years? Well, the debt will have grown to $10,000(1.06)^5 = $13,382.26 while the savings-account deposits will have accumulated to $Qs_{\overline{5}|.07}$ after five years, which is one year after the fifth $Q$ deposit. So $Q$ needs to be chosen so that

$$13382.26 = Qs_{\overline{5}|.07} = Q\frac{(1.07)^5 - 1}{.07/1.07} = 6.153Q,$$

which gives $Q = 2174.81$ – obviously a better deal for Schmidt.

The relations presented earlier among annuity-certain symbols followed easily from their definitions as representing values of the same monies but at different points in time. Some other
occasionally useful relations are less obvious. For example, in evaluating the annuity-certain-due symbol $\ddot{a}_n$ you can count the first payment at time-0 separately and then view the remaining $n - 1$ payments as an $(n - 1)$-year annuity-certain-immediate with present value $\ddot{a}_{n-1}$. This decomposition viewpoint produces $\ddot{a}_n = 1 + a_{n-1}$. Substituting $a_{n-1} = v \ddot{a}_{n-1}$ produces $\ddot{a}_n = 1 + v \ddot{a}_{n-1}$, just the sort of recursion you’ll see often in this book. Similar relations among the symbols $s$ and $\ddot{s}$ can be derived – some by multiplying the preceding by $1 + i$ to appropriate powers. The following theorem wraps up all these results in a single package.

**Theorem 1.23.** The following formulas hold for the values of level annuities-certain.

**Basic evaluations:**

$$\ddot{a}_n = \frac{1 - v^n}{d}, \quad \ddot{s}_n = \frac{(1 + i)^n - 1}{d},$$

$$a_n = \frac{1 - v^n}{i}, \quad s_n = \frac{(1 + i)^n - 1}{i}$$

**Time shifts:**

$$\ddot{a}_n = (1 + i)a_n = v^n \ddot{s}_n, \quad \ddot{s}_n = (1 + i)s_n = (1 + i)^n \ddot{a}_n$$

$$a_n = v \ddot{a}_n = v^n \ddot{s}_n, \quad s_n = v \ddot{s}_n = (1 + i)^n a_n$$

**Decompositions:**

$$\ddot{a}_n = 1 + a_{n-1}, \quad \ddot{s}_n = s_{n-1} - 1$$

$$a_n = \ddot{a}_{n-1} - 1, \quad s_n = 1 + \ddot{s}_{n-1}$$

**Recursions:**

$$\ddot{a}_n = 1 + v \ddot{a}_{n-1}, \quad \ddot{s}_n = (1 + i) + (1 + i)\ddot{s}_{n-1}$$

$$a_n = v + va_{n-1}, \quad s_n = 1 + (1 + i)s_{n-1}.$$
1.7 Amortization schedule

In this section, we review a method of repaying a loan, which is called the amortization method. In this method the borrower repays the lender by means of installment payments at periodic intervals. Clearly, the installment payments form an annuity whose present value is equal to the original amount of the loan.

In practice, it is very important to determine the amount of the outstanding loan balance. There are two approaches used in finding the amount of the outstanding loan balance; namely, the prospective method and the retrospective method. According to the prospective method, the outstanding loan balance at any point in time is equal to the present value at that date of the remaining payments. According to the retrospective method, the outstanding loan balance at any point in time is equal to the original amount of the loan accumulated to that date less the accumulated value at the date of all payments previously made. If the outstanding loan balance at time \( t \) for an original loan amount of \( a_{\bar{n}} \) is denoted by \( B_t \), then the prospective method gives

\[
B_t (= B_t^p) = a_{\bar{n}-t}, \quad (1.11)
\]

and the retrospective method gives

\[
B_t (= B_t^r) = a_{\bar{n}}(1 + i)^t - s_{\bar{n}t}. \quad (1.12)
\]

**Example 1.24.** A loan is being repaid by quarterly installments of \$1500 at the end of each quarter at 10% convertible quarterly. If the loan balance at the end of the first year is \$12,000, find the original loan balance. Answer to the nearest dollar.

*Using the retrospective method, we have*

\[
B_4 = B_0 (1 + j)^4 - 1500s_{\bar{4}|j}.
\]

where \( j \) is the quarterly rate of interest, which is \( \frac{10\%}{4} \). Solving for \( B_0 \), we obtain the original loan balance is \$16,514.
Example 1.25. A loan will be amortized at 16% yearly effective rate of interest by 6 years level payments of $600, followed by 5 yearly payments of $500. Find the balance on the debt an instant after the fourth payment is made.

Using the prospective method, we have

\[ 4 = 500a_{7|16} + 100a_{2|16} = \$2179.81. \]

An amortization schedule is a table which shows the division of each payment into principal and interest, together with the outstanding loan balance after each payment is made. Let \( R \) denote the installment payment in an \( n \)-PRDs loan \( L \) with the PRDly rate of \( i \). If we denote the amount of interest paid in the \( t^{th} \) installment by \( I_t \), and the amount of principal repaid in the same installment by \( P_t \). Then, for the amortization schedule, we have

\[ R = \frac{L}{a_{\overline{m}|i}} \]  

(1.13)

\[ I_t = iB_{t-1}, \text{ and } P_t = R - I_t. \]  

(1.14)

Example 1.26. A $50,000 loan at 24% nominal interest compounded semi-annually is amortized with level semi-annual payments for 10 years. Find the amount of interest in the 7th payment.

Let \( j \) be the semi-annually rate. Then \( j = \frac{24\%}{2} = .12. \) From (1.13), we have \( R = \frac{50,000}{a_{\overline{20}|.12}} \approx \$6693.98. \) The outstanding balance after the 6th payment is \( B_6 = Ra_{\overline{14}|.16} \approx \$44369, \) and thus the amount of interest paid in the 7th payment is \( I_7 = jB_6 = \$5324. \) Thus, the amount of principal paid in this period should be \( R - I_7 = 1369.97. \)

Problems.

1. A 7000 loan is being paid off with payments of 1000 at the end of each year for as long as necessary, plus a smaller
payment one year after the last regular payment. If $i = 0.11$ and the first payment is due one year after the loan is taken out, find the outstanding principal, $P$, immediately after the ninth payment.

2. A 1000 loan is repaid by annual payments of 150, plus a smaller final payment. If $i = .11$ and the first payment is one year after the time of the loan, find the amount of principal and interest contained in the third payment.

3. A loan is being paid by 20 annual payments. The first 5 installments are 300 each, the next 8 are 400 each, and the last 7 are 600 each. Assume $i = .14$.

   (a) Find the loan balance immediately after the tenth payment.

   (b) Divide the $11^{th}$ payment into principal and interest.

4. A loan is repaid by 20 equal annual payments at 11% effective. If the amount of principal in the $4^{th}$ payment is 150, find the amount of interest in the $12^{th}$ payment.

5. A loan is being paid with 30 equal annual installments. The principal portion of the $11^{th}$ payment is 247.13, and the interest portion is 352.87. Find $i$.

6. George was making annual payments of $X$ on a 16% 10-year loan. After making 4 payments, he renegotiates to pay off the debt in 3 more years with the lender being satisfied with 14% over the entire period. Find an expression for the new payment.

7. Harriet is repaying a loan with payments of 3000 at the end of every two years. If the interest in the $5^{th}$ installment is 1,982.31, find the amount of principal in the $8^{th}$ installment. Assume $i = .13$. 
1.8 Bonds

A bond is an interest-bearing security which promises to pay a stated amount(s) (redemption value) of money at some future date(s) (maturity date). Bonds may be classified as accumulation bonds and bonds with coupons. The coupons are periodic payments made by the issuer of the bond prior to its redemption. The coupon rate is customarily quoted as a nominal rate convertible semiannually, and it is applied to the face value (or called par value), which is stated on the front of the bond. An accumulation bond is one in which the redemption price includes the original loan plus all accumulated interest. We’ll discuss the bonds with coupons payable periodically, since accumulation bonds can easily be handled with compound interest method discussed in the earlier sections.

Let us recall some notation and review some basic terminology:

\[ F = \text{the face value or par value of the bond.} \]
\[ r = \text{the coupon rate per interest period.} \]

Thus the amount of each interest payment (coupon) is \( Fr \).

\[ C = \text{the redemption value of the bond. Often} F = C. \]
\[ i = \text{the yield rate per interest period.} \]
\[ n = \text{the number of interest periods until the redemption date.} \]
\[ P = \text{the purchase price of the bond to obtain yield rate} i. \]

In return for paying out \( P \) at time 0, the investor receives \( n \) coupons of value \( Fr \) each and a final payment of \( C \) at time \( n \). Hence, to obtain a yield rate \( i \), we have the purchase price

\[ P = (Fr)\alpha_{\overline{n| i}} + C(1 + i)^{-n}. \]

**Example 1.27.** A corporation decides to issue 15-year bonds with face amount of 1000 each. If interest payments are to be
made at the rate of 10% convertible semiannually, and if Joe is happy with a yield of 8% convertible semiannually, what should he pay for one of these bonds?

For these bonds we have \( F = C = 1000 \), \( n = 30 \), \( r = .05 \) and \( i = .04 \). Then \( P = 50a_{\overline{30}|.04} + 1000(\frac{1}{1.04})^{30} = 1172.92 \).

In the same sense that a loan has an outstanding balance at any point of time, we can talk about the book value of a bond at any time \( t \). If we are at a point in time where the \( t^{th} \) coupon has just been paid, then the book value at time \( t \) is the value of the remaining payments: \( n - t \) coupons and a payment of \( C \) at time \( n \). Hence, the book value is

\[
B_t = (Fr)a_{\overline{n-t}|i} + Cv^{n-t}.
\]

Obviously, the book value lies between \( P \) and \( C \) for time \( t \) between 0 and \( n \).

**Example 1.28.** Find the book value immediately after the payment of the 14\(^{th} \) coupon of a 10-year 1,000 par-value bond with semiannual coupons, if \( r = .05 \) and the yield rate is 12\% convertible semiannually.

We have \( F = C = 1000 \), \( n = 20 \), \( r = .05 \), and \( i = .06 \). At time \( t = 14 \) there are 6 coupons left, so the book value is

\[
(1000)(.05)a_{\overline{6}|i} + 1000(\frac{1}{1.06})^6 = 950.83.
\]

This is larger than \( P = 885.30 \), but smaller than \( C = 1000 \).

**Problems.**

1. Let \( B_t \) and \( B_{t+1} \) be the book values just after the \( t^{th} \) and \( (t+1)^{st} \) coupons are paid. Show that \( B_{t+1} = B_t(1+i) - Fr \).

2. A 10-year 1000 face value bond, redeemable at par, earns interest at 9\% convertible semiannually. Find the price to yield an investor 8\% convertible semiannually.
3. In problem 2, find the book value at the time just the 7th coupon has been paid.

4. One bond of face value 100 with 5% semiannual coupons costs 75.73. A similar bond with 8% semiannual coupons costs 112.14. Both are redeemable at par in n years and have the same yield rate i. Find i and n.

5. Find the price of a 1000 par-value 10 year bond which has quarterly 2% coupons and is bought to yield 9% per year convertible semiannually.

6. Find the price of a 1000 par-value 10 year bond which has semiannual coupons of 10 the first half-year, 20 the second half-year, ⋯, 200 the last half-year, bought to yield 9% effective per year.

7. A 1000 par value 11% bond has coupons payable on Jan 1 and July 1, and will be redeemed July 1, 1971. The bond was bought Jan. 1, 1969, to yield 12% convertible semiannually. Find the price.