Chapter 4 Probability and Counting Rules

4.1 Introduction

4.2 Sample Spaces and Probability

4.3 Addition Rule for Probability

4.4 Multiplication Rule and conditional probability

4.5 Counting Rules

4.7 More examples for probability and counting rules

Probability is a way of expressing knowledge or belief that an event will occur or has occurred. In mathematics the concept has been given an exact meaning in probability theory, that is used extensively in the areas of mathematics, statistics, finance, gambling, science, and philosophy to draw conclusions about the likelihood of potential events and the underlying mechanics of complex systems.

The probability of an event is a measure of the likelihood of the event occurring. Probability always falls in the closed interval from 0 to 1. An event with probability 0 is an event which will not occur, i.e., an impossible event. An event with probability 1 is an event will definitely occur, i.e., a certain event.

An experiment is an act or process of observation that leads to a single outcome (that typically cannot be predicted with certainty). Sometimes a basic outcome of an experiment is called a sample point. The
**sample space** \(S\) of a random phenomenon is the set of all possible outcomes. An **event** is any outcome or a set of a random phenomenon. That is, an event is a subset of the sample space. A **probability model** is a mathematical description of a random phenomenon consisting of two parts: a sample space \(S\) and a way of assigning probabilities to events. A **Venn diagram** is often a useful graphical representation of a sample space and events.

**Example.** Flip a coin twice. The possible outcomes (called the *sample space*) are: HH, HT, TH, TT. The probability of getting at least one H is \(3/4\). The probability of getting no H is \(1/4\).

**Definition.** A **random variable** is a variable whose value is a numerical outcome of a random phenomenon. The **probability distribution** of a random variable tells us what the possible values of the variable are and how probabilities are assigned to those values.

**Equally Likely Outcomes**

**Definition.** If a random phenomenon has \(k\) possible outcomes, all *equally likely*, then each individual outcome has probability \(1/k\). The probability of any event \(A\) is

\[
P(A) = \frac{\text{count of outcomes in } A}{\text{count of all possible outcomes}} = \frac{\text{count of outcomes in } A}{k}.
\]
Note. Some facts about probability:

- Any probability is a number between 0 and 1. That is, $0 \leq P(A) \leq 1$.
- The probability of the sample space is $P(S) = 1$, and all possible outcomes together must have probability 1.
- The probability that an event does not occur is 1 minus the probability that an event does occur. That is, $P(A^c) = 1 - P(A)$ for $A^c$ = the event of not $A$.

Example 4-35 on p.213: Four cards are drawn from an ordinary deck. Find the probability that at least one ace is drawn.

- If two events have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities. That is, $P(A \cup B) = P(A) + P(B)$ if two events $A$ and $B$ are disjoint.

In general, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

- Conditional Probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

Example 4-29.

Tree Diagram Example 4-31.

Definition. Two events $A$ and $B$ are said to be independent if $P(A \cap B) = P(A)P(B)$.

If $A$ and $B$ are independent, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$. 

3
Basic Counting Results (Combinatorics):

Multiplication principle. If a task has $k$ steps and there are $n_1$ ways to do step 1, $n_2$ ways to do step 2, \ldots and $n_k$ ways to do step $k$, then the number of ways to perform the task is the product

$$n_1 \cdot n_2 \cdots n_k$$

Example. If we make license plates with digits or letters, (a) how many such kind of license plates are there? (b) How many license plates are there for having three letters followed by three digits? (c) How many licence plates are there to have the letters MTSU followed by two digits?

- Tree diagrams are often helpful in counting

See Example 4-38 on p.218.

Factorial Notation: $n! = n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1$, $0! = 1$.

- number of permutations (ordered subsets): $nP_r = n(n - 1) \cdots (n - r + 1) = \frac{n!}{(n-r)!}$

Example. (1) How many letter arrangements are there from the word: MTSU?

(2) Example 4-45 Chairperson selection on p.223.

- number of combinations (subsets): $nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!}$

Example. (1) How many different tests can be made from a test bank of 20 questions if the test consists 5 questions?

(2) Example 4-49 on p.225 Committee Selection.
Inclusion/exclusion principle (simplest case): \( n(A \cup B) = n(A) + n(B) - n(A \cap B) \)

**Example.** How many outcomes are there for selecting a heart or a 10 if we draw a card from an ordinary deck?