Foundations of Math
Sample Problems (re Test 3)

1. Let \( R = \{(0,1), (1,2), (0,2), (2,3), (1,3), (3,0)\} \) be a relation on the set \( A = \{0, 1, 2, 3\} \).
   Is \( R \) transitive? Justify your answer.

2. Suppose \( R \) is a transitive relation on the set \( \{1, 2, 3, 4, 5\} \). If \( (1, 3) \in R \), \( (3, 5) \in R \), and \( (5,1) \in R \), what other ordered pairs must be elements of \( R \)?

3. Suppose \( R \) is a reflexive relation on the set \( \{1, 2, 3, 4, 5\} \).
   List all ordered pairs that must be elements of \( R \).

4. Suppose \( R \) is a symmetric relation on the set \( \{1, 2, 3, 4, 5\} \).
   If \( (1, 3) \in R \), what other ordered pair is an element of \( R \)?

5. Let \( R = \{(0,1), (0,2), (1,2), (2,1), (2,3), (1,3)\} \) be a relation on the set \( A = \{0, 1, 2, 3\} \).
   Is \( R \) antisymmetric? Justify.

6. Let \( R = \{(0,1), (0,2), (1,2), (1,3), (2,3), (3,0)\} \) be a relation on the set \( A = \{0, 1, 2, 3\} \).
   Find the inverse \( R^{-1} \).

7. Let \( \Pi = \{\{a, b\}, \{c, d\}\} \) be a partition of \( A = \{a, b, c, d\} \). Find the equivalence relation \( R \)
   induced by \( \Pi \).
   \[ R = A/\Pi = \{(a,a), \} \]

8. Let \( A = \{2, 4, 6, 8\} \). Let \( R \) denote the partial order of \( A \) defined by
   \( xRy \) if and only if \( x \) divides \( y \).
   Find \( R \).

9. Define the relation \( R \) on \( \mathbb{N} = \{0, 1, 2, \ldots\} \) by \( xRy \) if and only if \( \frac{x-y}{3} \) is an integer.
   Find the equivalence class \([2]_R\).
   \[ [2]_R = \{ \ldots \} \]

10. Draw the Hasse diagram for the following partial order:
    \( R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\} \)

11. Let \( D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\} \) be the set of divisors of 30. Define a partial order by
    \( x \preceq y \) if and only if \( x \) divides \( y \). Draw the Hasse diagram for the poset \((D_{30}, \preceq\)).
12. Find the partial order \( R \) that corresponds to the following Hasse diagram.

\[
\begin{array}{c}
\circlearrowleft \\
\circlearrowright \\
\end{array}
\]

\[ R = \{ \}

13. Let \( A = \{1, 2, 4, 6, 8\} \). Define a partial order by \( x \leq y \) if and only if \( x \) divides \( y \). List all the maximal elements of poset \( (A, \leq) \).

14. Let \( A = \{1, 2, 4, 6, 8\} \). Define a partial order by \( x \leq y \) if and only if \( x \) divides \( y \). For poset \( (A, \leq) \), find the least upper bound of the set \( \{2, 4\} \), if it exists.

15. Let \( A = \mathbb{R} \times \mathbb{R} \). Define the relation \( R \) by \((x, y)R(v, w)\) iff \( x^2 + y^2 = v^2 + w^2 \). Is \( R \) an equivalence relation? Justify your answer.

16. Let \( R = \{(0, 1), (1, 2), (0, 2)\} \) be a relation on the set \( A = \{0, 1, 2\} \). Find \( R \circ R^{-1} \).

17. Let \( R \) be a relation on some nonempty set \( A \) and suppose \( R \circ R \subseteq R \). By the definition of composition, if \((x, y) \in R \) and \((y, z) \in R \), then \((x, z) \in R \circ R \). But, since \( R \circ R \subseteq R \), if \((x, z) \in R \circ R \), then \((x, z) \in R \). Hence, we have proven that . . .

(a) If \( R \) is a relation on some nonempty set and \( R \circ R \subseteq R \), then \( R \) is transitive.
(b) If \( R \) is a relation on some nonempty set and \( R \) is transitive, then \( R \circ R \subseteq R \).
(c) If \( R \) is a relation on some nonempty set and \( R \) is transitive, then \( R \subseteq R \circ R \).
(d) If \( R \) is a relation on some nonempty set and \( R \) is transitive, then \( R \circ R = R \).
(e) If \( R \) is a relation on some nonempty set, then \( R \circ R \subseteq R \) if and only if \( R \) is transitive.

18. Let \( R = A \times A \) where \( A = \{1, 2, 3\} \).

<table>
<thead>
<tr>
<th>Property</th>
<th>Yes</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R ) is reflexive on ( A ).</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>( R ) is symmetric.</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>( R ) is antisymmetric.</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>( R ) is transitive.</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>
19. Complete the proof below that \( R \circ (S \cup T) \subseteq (R \circ S) \cup (R \circ T) \).

Let \((x, y) \in R \circ (S \cup T)\). Then \((x, z) \in S \cup T\) and \((z, y) \in R\) for some \(z\).
Since \((x, z) \in S \cup T\), we have \((x, z) \in S\) or ...

20. Let \(A = \{0, 1, 2, 3, 4\}\). Let \(P(A)\) denote the power set of \(A\). What is the cardinality of \(P(A)\)?

21. Let \(A = \{0, 1, 2, 3, 4\}\). Let \(P(A)\) denote the power set of \(A\). For all \(B, C \in P(A)\), define the relation \(R\) so that \(BRC\) if and only if \(\text{Card}(B) = \text{Card}(C)\). Is \(R\) an equivalence relation on \(P(A)\)? Justify your answer.

22. Let \(\mathbb{Z}\) denote the set of integers. Find two subsets \(A\) and \(B\) of \(\mathbb{Z}\) such that (i) \(A \subseteq B\) and (ii) \(A \cong B\).

23. Let \(P(\mathbb{N})\) denote the power set of the positive natural numbers. Is \(P(\mathbb{N}) \approx [0,1]\)? Why?

24. If \(A\) and \(B\) are finite sets, then \(\text{Card}(A \cup B) = \text{Card}(A) + \text{Card}(B)\). True False

25. If \(A\) and \(B\) are finite sets, then \(\text{Card}(A \times B) = \text{Card}(A) \cdot \text{Card}(B)\). True False

26. \(\mathbb{N} \approx \mathbb{N} \times \mathbb{N}\) True False

27. Let \(\mathbb{Q}^+\) denote the nonnegative rational numbers. Then \(\mathbb{N} \approx \mathbb{Q}^+\). True False

28. If \(A\) is countably infinite (denumerable) and \(B\) is countable infinite (denumerable), then \(A \cup B\) is countably infinite (denumerable). True False

29. Sixteen positive integers from the set \(\{1, 2, 3, \ldots, 30\}\) are chosen at random. Among the chosen integers the sum of two of them is equal to 31. [True] False

Let \(S = \{1, 2, 3, \ldots, 30\}\) and let \(R\) denote the set of 16 chosen integers from \(S\).
Define function \(f : R \to S\) by \(f(r) = 31 - r\). If we can show that there exists an \(r \in R\) such that \(f(r) \in R\), we are done since \(r + f(r) = 31\) and \(r \neq f(r)\).

Now note that \(f\) is one-to-one. Hence \(\text{Card}[f(R)] = \text{Card}[R] = 16\). Also \(\text{Card}[R \cup f(R)] \leq 30\). Thus

\[
\text{Card}[R \cap f(R)] = \text{Card}[R] + \text{Card}[f(R)] - \text{Card}[R \cup f(R)] \\
\geq 16 + 16 - 30 \\
\geq 2.
\]

Since \(R \cap f(R)\) is nonempty, there exists an \(x \in R \cap f(R)\). Hence \(x \in R\) and \(x = f(r) \in R\) for some \(r \in R\) with \(r \neq x\). Thus \(r + f(r) = 31\) and we are done.
30. Let $A = \{0, 1, 2\}$. Let $P(S)$ denote the power set of $S$. What is the cardinality of $P(P(A))$?

31. Let $A = \{1, 2, 4\}$ and $B = \{1, 2, 3, 4, 5\}$. Find the number of injective functions from $A$ to $B$.

32. If all the people in a group of 10 people shake hands with one another, how many handshakes occur?

33. If sets $A$ and $B$ both have an infinite number of elements, then $A \approx B$. True False