1. Find gcd(378, 72) using the Euclidean algorithm.

\[ 378 = 72(5) + 18 \]
\[ 72 = 18(4) + 0 \]

answer: 18

2. Find lcm(378, 72).

\[ \text{lcm}(378, 72) = \frac{378(72)}{\text{gcd}(378, 72)} = \frac{378(72)}{18} = 1512 \]

answer: 1512

3. Find integers \( x \) and \( y \) such that \( 17x + 23y = 1 \).

\[ 23 = 17(1) + 6 \]
\[ 17 = 6(2) + 5 \]
\[ 6 = 5(1) + 1 \implies 1 = 6 - 5 = (23 - 17) - (17 - 6(2)) \]
\[ = 23 - 2(17) + (23 - 17)(2) \]
\[ = ( -4)17 + 3(23) \]

\[ x = -4, \ y = 3 \]

4. Suppose that you have two unmarked beakers that will hold exactly 3 cc and 17 cc, respectively. Describe how you would measure out exactly 1 cc of liquid using only the two beakers.

<table>
<thead>
<tr>
<th>( B_3 ) Operation</th>
<th>( B_{17} ) Operation</th>
<th>Amount in ( B_3 )</th>
<th>Amount in ( B_{17} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fill ( B_3 ), Pour all into ( B_{17} )</td>
<td></td>
<td>3 cc</td>
<td></td>
</tr>
<tr>
<td>Fill ( B_3 ), Pour all into ( B_{17} )</td>
<td></td>
<td>6 cc</td>
<td></td>
</tr>
<tr>
<td>Fill ( B_3 ), Pour all into ( B_{17} )</td>
<td></td>
<td>9 cc</td>
<td></td>
</tr>
<tr>
<td>Fill ( B_3 ), Pour all into ( B_{17} )</td>
<td></td>
<td>12 cc</td>
<td></td>
</tr>
<tr>
<td>Fill ( B_3 ), Pour all into ( B_{17} )</td>
<td></td>
<td>15 cc</td>
<td></td>
</tr>
<tr>
<td>Fill ( B_3 ), Pour into ( B_{17} ) \ until full</td>
<td></td>
<td>1 cc</td>
<td>17 cc</td>
</tr>
<tr>
<td>Empty ( B_{17} )</td>
<td></td>
<td>1 cc</td>
<td>0 cc</td>
</tr>
</tbody>
</table>

*************** *********** ********** **********
5. Prove that there is an infinite number of prime numbers.

**Proof** (by contradiction). Assume there is a finite number of primes, say \( m \) in number. If \( p_i \) denotes the \( i \)-th smallest prime, then the product of all primes is \( p_1 p_2 \cdots p_m \).

Consider the integer \( b = p_1 p_2 \cdots p_m + 1 \). Since \( b \) is larger than the largest prime \( p_m \), \( b \) is not a prime and thus is divisible some prime \( p_k \in \{ p_1, \ldots, p_m \} \). But \( p_k \) does not divide \( b \) since the remainder is 1.

6. Prove that 6 divides \( 25^n - 1 \) for \( n = 0, 1, 2, \ldots \).

**Proof.** Since \( 25 = 6(4) + 1 \), \( 25 = 1 \pmod{6} \). Hence \( 25^n = 1^n \pmod{6} \), and thus 6 divides \( 25^n - 1 \).

7. Prove that \( \sqrt{7} \) is irrational.

**Proof** (by contradiction). Assume \( \sqrt{7} \) is rational. Then \( \sqrt{7} = \frac{m}{n} \) for some positive integers \( m \) and \( n \). Hence \( 7n^2 = m^2 \). However, 7 is a factor of \( 7n^2 \) with odd multiplicity and 7 is a factor of \( m^2 \) with even multiplicity, contradicting the Prime Factorization Theorem. Therefore \( \sqrt{7} \) is irrational.
8. On day 1 you put 1 penny in your huge, heretofore empty, piggy bank. On each subsequent day you double your previous day's deposit into the bank, that is, 2 pennies are deposited on day 2, 4 pennies are deposited on day 3, etc. You do this for 88 days. Prove that your total savings after 88 days can be divided into 5 equal piles of pennies.

**Proof.** Let \( T \) denote your total savings in pennies. Then \( T = 1 + 2 + \ldots + 2^{87} = 2^{88} - 1 \). But \( 2^4 = 1 + 5(3) \) which implies \( 2^4 \equiv 1 \pmod{5} \). Thus \( 2^{88} = (2^4)^{22} \equiv 1^{22} \pmod{5} \). Hence 5 divides \( 2^{88} - 1 \).

9. For positive integer \( n \), if \( 2^n - 1 \) is prime, then \( n \) is prime. 

If \( n \) is not prime, then \( n = rs \) for some positive integers greater than 1.

Hence \( 2^n - 1 = (2^r)^s - 1 = (2^r - 1)(1 + 2^r + 2^{2r} + \ldots + 2^{(s-1)r}) \),

which is clearly not prime.

10. There exists two integers \( m \) and \( n \) such that \( 31m + 18n = 1 \). 

The statement is true since \( \gcd(31, 18) = 1 \).

11. \( (12345)^{100} \equiv 0 \pmod{3} \) 

Note the \( 1 + 2 + 3 + 4 + 5 = 15 \) and 15 is divisible by 3. hence 3 divides 12345 and 3 divides \( (12345)^{100} \).

12. If \( 7n = 7m \pmod{5} \), then \( n = m \pmod{5} \). 

Since \( \gcd(7, 5) = 1 \), the 7 can be cancelled.

13. Without a calculator, use a divisibility rule to prove that 7 divides 672672.

7 divides 672 since 7 divides \( 67 - 2(2) \). Hence 7 divides 672672.