1. Find gcd(444, 36) using the Euclidean algorithm.

Since \( 444 = 36(12) + 12 \)
and \( 36 = 12(3) \),
gcd(444, 36) = 12.

\[ \text{answer: 12} \]

2. Find lcm(444, 36).

Since \( \text{lcm}(444, 36) = \frac{444(36)}{\text{gcd}(444, 36)} = \frac{444(36)}{12} = 1332 \)

\[ \text{answer: 1332} \]

3. Find gcd(35, 49) and integers \( x \) and \( y \) such that \( 35x + 49y = \text{gcd}(35, 49) \).

\[ 49 = 35(1) + 14 \]
\[ 35 = 14(2) + 7 \]
\[ 14 = 7(2) \quad \Rightarrow \quad \text{gcd}(35, 49) = 7 \]

and
\[ 7 = 35 - 14(2) \]
\[ = 35 - (49 - 35(1))(2) \]
\[ = 3(35) - 2(49) \]
\[ x = 3 \quad , \quad y = -2 \]

4. Without a calculator, prove that 7 divides 11111.

We have \( 11111 - 2 = 11109 \)
and \( 1110 - 18 = 1092 \)
and \( 109 - 4 = 105 \)
and \( 10 - 10 = 0. \)

7 divides 0 \( \Rightarrow \) 7 divides 105
\[ \Rightarrow \) 7 divides 1092
\[ \Rightarrow \) 7 divides 1109
\[ \Rightarrow \) 7 divides 11111
5. Suppose that you have two unmarked beakers that will hold exactly 5 cc and 7 cc, respectively. Describe how you would measure out exactly 1 cc of liquid using only the two beakers.

**Solution.** Since the \( \gcd(5, 7) = 1 \), there is a solution. Furthermore, since \( 3(5) - 2(7) = 1 \), the 5 cc beaker must be filled 3 times and the 7 cc beaker must be filled 2 times (by transfers) as shown in the solution table below.

<table>
<thead>
<tr>
<th>( B_5 ) Operation</th>
<th>( B_7 ) Operation</th>
<th>Amount in ( B_5 )</th>
<th>Amount in ( B_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fill ( B_5 )</td>
<td></td>
<td>5 cc</td>
<td>0</td>
</tr>
<tr>
<td>Pour all into ( B_7 )</td>
<td></td>
<td>0</td>
<td>5 cc</td>
</tr>
<tr>
<td>Fill ( B_5 )</td>
<td></td>
<td>5 cc</td>
<td>5 cc</td>
</tr>
<tr>
<td>Pour 2 cc into ( B_7 )</td>
<td></td>
<td>3 cc</td>
<td>7 cc</td>
</tr>
<tr>
<td>Empty ( B_7 )</td>
<td></td>
<td>3 cc</td>
<td>0 cc</td>
</tr>
<tr>
<td>Pour 3 cc into ( B_7 )</td>
<td></td>
<td>0</td>
<td>3 cc</td>
</tr>
<tr>
<td>Fill ( B_5 )</td>
<td></td>
<td>5 cc</td>
<td>3 cc</td>
</tr>
<tr>
<td>Pour 4 cc into ( B_5 )</td>
<td></td>
<td>[1 \text{cc}]</td>
<td>7 cc</td>
</tr>
</tbody>
</table>

6. Prove that there is an infinite number of prime numbers.

Proof (by contradiction). Assume there is a finite number \( m \) of primes. Let \( p_i \) denote the \( i \)-th smallest prime and let \( S = \{p_1, p_2, \ldots, p_m\} \). The integer \( x \) defined by \( x = p_1 p_2 \cdots p_m + 1 \) is larger than all the primes and hence must be composite. Therefore, by the prime factorization theorem, some \( p_j \in S \) must divide \( x \). But, by the definition of \( x \), \( x = 1 \pmod{p_j} \), which contradicts that \( p_j \) divides \( x \).

7. For positive integer \( n \), if \( 2^n - 1 \) is prime, then \( n \) is prime. **True**  
   False

It clearly holds for \( n = 3 \). For \( n > 3 \), we prove the contrapositive. Assume \( n \) is composite, that is, \( n = rs \) for some integers \( r \) and \( s \) both greater than 1. Then

\[
2^n - 1 = 2^{rs} - 1 = (2^r)^s - 1 = (2^r - 1) \sum_{k=0}^{s-1} (2r)^k,
\]

using the formula \( a^m - b^m = (a - b) \sum_{k=0}^{m-1} a^k b^{m-1-k} \) with \( a = 2^r, b = 1, \) and \( m = s \).

Since both \( (2^r - 1) \) and \( \sum_{k=0}^{s-1} (2r)^k \) are integers greater than 1, their product \( 2^n - 1 \) is composite.
8. There exists two integers $m$ and $n$ such that $30m + 18n = 1$. $\quad$ True  False

Any linear combination of 30 and 18 will be even.

9. $12345 = 0 \pmod{3}$ $\quad$ True  False

$1 + 2 + 3 + 4 + 5 = 15$ and 15 is divisible by 3

10. If $p$ is a prime and $xy = 0 \pmod{p}$, then $x = 0 \pmod{p}$ or $y = 0 \pmod{p}$. $\quad$ True  False

If $p$ is a prime factor of $xy$, then $p$ must appear in the prime factorization of $x$ or in the prime factorization of $y$, possibly in both.

11. Prove that if 3 divides $n^2$, then 9 divides $n^2$.

Assume 3 divides $n^2$. By the Prime Factorization Theorem, $n$ has a unique prime factorization and thus $n^2$ must have each prime factor of $n$ occurring an even number of times. Therefore the factor 3 has multiplicity $2k$ for some positive integer $k$, that is, $3^{2k}$ divides $n^2$. Since $3^{2k} = 9^k$, which is a multiple of 9, 9 divides $n^2$.

12. Prove that $\sqrt{3}$ is irrational.

Proof (by contradiction). Assume otherwise. Then $\sqrt{3} = \frac{m}{n}$ for some positive integers $m$ and $n$. Hence $3n^2 = m^2$. By the prime factorization theorem, 3 must be a factor with the same multiplicity for both $3n^2$ and $m^2$. But 3 is a factor of $3n^2$ an odd number of times and 3 is a factor of $m^2$ an even number of times, contradicting the prime factorization theorem.

13. On day 1 you put 1 penny in your huge, heretofore empty, piggy bank. On each subsequent day you double your previous day's deposit into the bank, that is, 2 pennies are deposited on day 2, 4 pennies are deposited on day 3, etc. You do this for 99 days. Prove that your total savings after 99 days can be divided into 7 equal piles of pennies.

Proof. After 99 days, you will have saved $\sum_{k=0}^{98} 2^k$ pennies. But $\sum_{k=0}^{98} 2^k = 2^{99} - 1$. Since $2^3 = 1 \pmod{7}$, $(2^3)^{33} = 1^{33} \pmod{7} = 1 \pmod{7}$. Therefore 7 divides $2^{99} - 1$. 