Chapter 4 — Cartesian Products and Functions

4.1 Product Sets

• An ordered pair is a pairing of a first coordinate and a second coordinate. Example. \((-3, 2)\) is an ordered pair.

• Equality of ordered pairs: \((x, y) = (z, w)\) iff \(x = z\) and \(y = w\)

  Example. \((x, y) = (3, 4)\) iff \(x = 3\) and \(y = 4\)

  Example. \((3, 4) \neq (4, 3)\)

• The Cartesian product of sets \(A\) and \(B\) is given by

  \[ A \times B = \{(x, y) : x \in A \text{ and } y \in B\}. \]

  Note that the Cartesian product of two sets is a set of ordered pairs, but not all sets of ordered pairs are Cartesian products.

  Example. Let \(A = \{1, 2\}\) and let \(B = \{0, 4\}\). Then \(A \times B = \{(1, 0), (1, 4), (2, 0), (2, 4)\}\).

  Example. \(\{(1, 2), (2, 3)\}\) is not a Cartesian product.

• \(A \times B = \emptyset\) iff \(A = \emptyset\) or \(B = \emptyset\)

• Let \(R\) denote the set of real numbers. The Cartesian plane is \(R \times R\).

  The \(x\)-axis is \(R \times \{0\}\). The \(y\)-axis is \(\{0\} \times R\).

• Any set of ordered pairs in the plane is a graph.

  Example. The verticle line given by equation \(x = 5\) is the Cartesian product \(\{5\} \times R\).

• Theorem. If \(A \subseteq D\) and \(B \subseteq E\), then \(A \times B \subseteq D \times E\).

  Proof. Assume \(A \subseteq D\) and \(B \subseteq E\). Let \((x, y) \in A \times B\). Then \(x \in A\) and \(y \in B\).

  Furthermore \(x \in D\) since \(A \subseteq D\), and \(y \in E\) since \(B \subseteq E\). But \(x \in D\) and \(y \in E\) implies that \((x, y) \in D \times E\).

• Theorem. \((A \cap B) \times C = (A \times C) \cap (B \times C)\)

• Question. \(C \times (A \cap B) \, ? \, (C \times A) \cap (C \times B)\)
4.2 Functions

• A function $f$ is a set of ordered pairs such that whenever $(x, u) \in f$ and $(x, v) \in f$, then $u = v$.

  - domain of $f = \text{dom}(f) = \{x : x \text{ is the first coordinate of some pair in } f\}$
  - range of $f = \text{ran}(f) = \{y : y \text{ is the second coordinate of some pair in } f\}$

• We say that $f$ is a function from $D$ to $C$ if $f$ is a function with $\text{dom}(f) = D$ and $\text{ran}(f) \subseteq C$. The set $C$ is called the codomain of $f$.

• If $(x, w) \in f$, then we write $w = f(x)$, which is read as the value of $f$ at $x$ or the image of $x$ under $f$.

• Two functions $f$ and $g$ are equal iff $\text{dom}(f) = \text{dom}(g)$ and $f(x) = g(x)$ for all $x \in \text{dom}(f)$.

- The floor function: $\lfloor x \rfloor = n$ when $n \leq x < n + 1$.
- The ceiling function: $\lceil x \rceil = n$ when $n - 1 < x \leq n$

• Let $A \subseteq D$, $f$ be a function from $D$ to $C$, and $g$ be a function from $A$ to $C$. Then $g$ is a restriction of $f$ if $g \subseteq f$. In this case we write $g = f_{\mid A}$.

• Identity function: $1_A : A \rightarrow A$ is defined by $1_A(x) = x$ for all $x \in A$.

• A real-valued function has codomain equal to $\mathbb{R}$, the set of real numbers.

• Vertical-line test (VLT): any vertical line intersects the graph of a function in at most one point.
  - Example. The function $f$ defined by $f(x) = x^2$ passes the VLT.
  - Example. The graph of the equation $x^2 + y^2 = 4$ fails the vertical line test.

• When a real-valued function $f$ is specified by a rule, the set of all real numbers for which the rule is meaningful is called the domain of definition for $f$.

  - Example. Consider the function $y = f(x) = \frac{1}{\sqrt{2-x}}$. The domain of definition is all real numbers such that $2 - x > 0$, that is, the $\text{dom}(f) = \{x \in \mathbb{R} : x < 2\}$.

• A sequence is a function $s$ with $\text{dom}(s) = \{m, m + 1, m + 2, \ldots\} \subseteq \mathbb{Z}$.
  - Notation: $s(n) = s_n$.

  - Example (geometric sequence). Let $s$ be defined by $s(n) = 3(2)^{-n}$ for $n = 0, 1, 2, \ldots$.
  - Example (geometric sequence). Let $s$ be defined by $s(n) = 3^n$ for $n = 0, 1, 2, \ldots$.
  - Example (arithmetic (linear) sequence). Let $s$ be defined by $s(n) = 4n - 3$ for $n = 1, 2, \ldots$.
  - Example (Fibonacci sequence). Let $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n - 1) + F(n - 2)$ for $n = 3, 4, \ldots$. 
• A function $f : A \to A$ is called a **unary operation** on $A$.
Set $B$ is **closed under** $f$ if $f(x) \in B$ for all $x \in B$.

Example. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ is a unary operation on $\mathbb{N}$.
The set of all even natural numbers is closed under $f$.
The set of all odd natural numbers is not closed under $f$.

• A function $g : A \times A \to A$ is called a **binary operation** on $A$.
Set $C$ is **closed under** $g$ if $g(x, y) \in C$ for all $x$ and $y \in C$.

Example. Addition is a binary operation on the reals.
Example. $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $g(m, n) = (m + 2)^n$ is a binary operation on $\mathbb{N}$
The set of even natural numbers is closed under $g$.
The set of odd natural numbers is closed under $g$.
The set of primes is not closed under $g$.

4.3 Compositions, Bijections, and Inverse Functions

• **Defn.** A function $f$ is **one-to-one** (or injective) if, for all $x, z \in \text{dom}(f)$, $f(x) = f(z)$ implies $x = z$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Function $f$ is injective since $x^3 = y^3$ implies $x = y$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Function $f$ is not injective since $x^2 = y^2$ does not imply $x = y$. Note that $(-1)^2 = 1^2$ but $-1 \neq 1$.

[No two distinct elements in the domain get mapped to the same element in the codomain.]

• **Horizontal-line test.** A function is injective if its graph passes the horizontal line test, that is, any horizontal line intersects the graph at one point at most.

• **Defn.** A function $f : D \to C$ is **increasing** if $x > y$ implies $f(x) > f(y)$ for all $x, y \in D$.

• **Defn.** A function $f : D \to C$ is **decreasing** if $x > y$ implies $f(y) > f(x)$ for all $x, y \in D$.

• All increasing functions are one-to-one.

Proof. Let $f : D \to C$ be an increasing function. Let $x, y \in D$ with $x \neq y$. Without loss of generality, assume $x > y$. Since $f$ is increasing, $f(x) > f(y)$ and thus $f(x) \neq f(y)$. Since $x \neq y$ implies $f(x) \neq f(y)$, $f$ is one-to-one. □
• All decreasing functions are one-to-one.

• Defn. A function \( f \) is onto (or surjective) if, for every element \( y \) in the codomain \( C \) of \( f \), there is at least one \( x \) in the domain \( D \) of \( f \) such that \( f(x) = y \).

\[ \text{Onto : Every element in the codomain has a preimage.} \]
\[ \text{Onto : The range of } f \text{ equals the codomain of } f. \]

• Defn. A function \( f \) is bijective if and only if \( f \) is one-to-one and onto.

• Defn. A permutation is a bijective function from a finite set \( A \) to itself.

• Defn. Composition of functions. Let \( f : D \to A \) and \( g : B \to C \), where \( \text{ran}(f) \subseteq B \). The \textit{composition} \( g \circ f : D \to C \) is defined by

\[ (g \circ f)(x) = g(f(x)) \]

for all \( x \in D \).

• Defn. The inverse of a function. Let \( f : D \to C \) be a function. Then \( g : C \to D \) is the inverse of \( f \) if \( g \circ f = 1_D \) and \( f \circ g = 1_C \).

When \( g \) is the inverse of \( f \), we write \( g = f^{-1} \).

• If \( f \) has an inverse, then \( f \) is bijective.

• If \( g : D \to A \) is onto and \( f : A \to C \) is onto, then \( f \circ g : D \to C \) is onto.

• If \( g : D \to A \) is 1-1 and \( f : A \to C \) is 1-1, then \( f \circ g : D \to C \) is 1-1.

Proof. Assume \( f \) and \( g \) are one-to-one. Suppose \((f \circ g)(x) = (f \circ g)(y)\). Then \( f(g(x)) = f(g(y)) \). Since \( f \) is one-to-one, \( g(x) = g(y) \). Furthermore, since \( g \) is one-to-one, we have \( x = y \).

• If \( f : D \to C \) and \( g : C \to D \) are bijective, then \((g \circ f)^{-1} = f^{-1} \circ g^{-1}\).

### 4.4 Images and Inverse Images of Sets

• Let \( f : D \to C \) and let \( A \subseteq D \). Then the \textit{image} of \( A \) under \( f \) is

\[ f[A] = \{ f(x) : x \in A \}. \]

• Let \( f : D \to C \) and let \( B \subseteq C \). Then the \textit{inverse image} of \( B \) under \( f \) is

\[ f^{-1}[B] = \{ x : x \in D \text{ and } f(x) \in B \} \]
• By definition of the image of $A$ under $f$, $x \in A \Rightarrow f(x) \in f[A]$.

• By definition of the inverse image of $B$ under $f$, $f(x) \in B \Rightarrow x \in f^{-1}[B]$.

• Note that $f(x) \in f[A]$ does not imply $x \in A$.

• Note that $x \in f^{-1}[B]$ implies $f(x) \in B$. (See Fact 6 below).

**FACTS**

1. If $A \subseteq B$, then $f[A] \subseteq f[B]$.
2. If $E \subseteq F$, then $f^{-1}[E] \subseteq f^{-1}[F]$.
3. $f[A \cap B] \subseteq f[A] \cap f[B]$.
5. $f[f^{-1}[E]] \subseteq E$.
6. $x \in f^{-1}[B]$ implies $f(x) \in B$.
7. If $f$ is onto, then $f[f^{-1}[E]] = E$.
8. $A \subseteq f^{-1}[f[A]]$.
9. If $f$ is one-to-one, $A = f^{-1}[f[A]]$.
10. If $f$ is one-to-one, $f[A \cap B] = f[A] \cap f[B]$.

**Proof of Fact 1.** Assume $A \subseteq B$. Let $y \in f[A]$. Then there exists $x \in A$ such that $y = f(x)$. But $A \subseteq B$ implies $x \in B$, which in turn implies $f(x) \in f[B]$. Since $y = f(x)$, we have $y \in f[B]$.

**Proof of Fact 3.** Let $y \in f[A \cap B]$. Then there exists $x \in A \cap B$ such that $y = f(x)$. Since $x \in A$ and $x \in B$, $f(x) \in f[A]$ and $f(x) \in f[B]$. Thus $f(x) \in f[A] \cap f[B]$.

**Proof of Fact 5.** Let $y \in f[f^{-1}[E]]$. Then there exists $x \in f^{-1}[E]$ such that $y = f(x)$. But $x \in f^{-1}[E]$ implies $f(x) \in E$. Hence $y \in E$ since $y = f(x)$.

**Proof of Fact 7.** Assume $f$ is onto. Let $y \in E$. Since $f$ is onto, $y = f(x)$ for some $x$ in the domain of $f$. Hence $x \in f^{-1}[E]$, which in turn implies $f(x) \in f[f^{-1}[E]]$. Therefore $E \subseteq f[f^{-1}[E]]$. Furthermore, by fact 5, $f[f^{-1}[E]] \subseteq E$. Thus $f[f^{-1}[E]] = E$.

**Proof of Fact 9.** Since $A \subseteq f^{-1}[f[A]]$ by Fact 8, we need only show $f^{-1}[f[A]] \subseteq A$.

Assume $f$ is one-to-one. Let $x \in f^{-1}[f[A]]$. Then $f(x) \in f[A]$. Hence there exists $w \in A$ such that $f(w) = f(x)$. Since $f$ is one-to-one, we must $w = x$. Thus $x \in A$. 
