1. Let \( R = \{(0, 1), (1, 2), (0, 2), (1, 1)\} \) be a relation on the set \( A = \{0, 1, 2\} \).
Is \( R \) antisymmetric? Justify your answer.

\[ R \text{ is antisymmetric since } (1, 0), (2, 1) \text{ and } (2, 0) \notin R. \]

2. Let \( A = \{1, 2\}, B = \{2, 3\}, C = \{3, 4\}, \) and \( D = \{4, 5\}. \)
Find \((A \cup B) \times (C \cap D)\).

Since \( A \cup B = \{1, 2, 3\} \) and \( C \cap D = \{4\}, \)
\((A \cup B) \times (C \cap D) = \{(1, 4), (2, 4), (3, 4)\}\)

3. Let \( f = \{(1, 4), (2, 1), (3, 2), (4, 3)\} \). Find \( f^{-1} \).

\[ f^{-1} = \{(4, 1), (1, 2), (2, 3), (3, 4)\} \]

4. Let \( A = \{1, 2, 5, 10\} \). Let \( R \) denote the partial order of \( A \) defined by \( xRy \) if and only if \( x \) divides \( y \). Find \( R \).

\[ R = \{(1, 1), (1, 2), (1, 5), (1, 10), (2, 2), (2, 10), (5, 5), (5, 10), (10, 10)\} \]

5. Let \( A = \mathbb{R} \times \mathbb{R} \). Define the relation \( R \) by \((x, y)R(v, w)\) \iff \(|x| + |y| = |v| + |w|\).
Complete the proof that \( R \) is an equivalence relation.

\[ R \text{ is reflexive since } |x| + |y| = |x| + |y| \text{ for every } (x, y) \in \mathbb{R} \times \mathbb{R}. \]

\[ R \text{ is symmetric since } |x| + |y| = |v| + |w| \text{ implies } |v| + |w| = |x| + |y|. \]

\[ R \text{ is transitive since } |x| + |y| = |v| + |w| \text{ and } |v| + |w| = |r| + |s| \text{ implies } |x| + |y| = |r| + |s|. \]
6. Draw the Hasse diagram for the following partial order. Let \( A = \{1, 2, 3, 4, 6, 12\} \). Let \( R \) denote the partial order of \( A \) defined by \( x R y \) if and only if \( x \) divides \( y \).

7. Let \( R = \{(0, 2), (1, 0), (2, 1)\} \) be a relation on the set \( A = \{0, 1, 2\} \). Find \( R \circ R^{-1} \).

8. Let \( \mathbb{R}^+ \) denote the set of nonnegative reals. If \( f: \mathbb{R}^+ \to \mathbb{R} \) is defined by \( f(x) = 2 \sqrt{x} + 1 \). Find \( f^{-1}(x) \).

9. Let \( f: \mathbb{R}^+ \to \mathbb{R} \) be defined by \( f(x) = 2 \sqrt{x} + 1 \). Find \( f[\mathbb{R}^+] \), the range of \( f \).
10. Let \( f : \mathbb{R}^+ \to \mathbb{R} \) be defined by \( f(x) = 2\sqrt{x} + 1 \). Let \( A = [3, 5] \). Find \( f^{-1}[A] \).

Since \( f^{-1} \) is increasing, \( f^{-1}[A] = [f^{-1}(3), f^{-1}(5)] = [1, 4] \).

\[ f^{-1}[A] = [1, 4]. \]

11. Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 + 9 \), and let \( g : \mathbb{R}^+ \to \mathbb{R} \) be defined by \( g(x) = \frac{1}{\sqrt{x}} - x \). Find \( (g \circ f)(4) \).

Since \( f(4) = 25 \) and \( g(25) = \frac{1}{5} - 25 = -\frac{124}{5} \).

\[ (g \circ f)(4) = -\frac{124}{5} \]

12. (True/False). Let \( A = \{1, 2, 3\} \) and \( B = \{1, 2\} \). The function \( f = \{(1, 1), (2, 2), (3, 1)\} \) is a one-to-one function from \( A \) to \( B \).

Since \( f(1) = f(3) \), \( f \) is not one-to-one. True \hspace{1cm} False

13. The set of positive rational numbers is closed under division.

Let \( a, b, c, d \in \mathbb{Z}^+ \), the set of positive integers.
Then \( \frac{a}{b} = \frac{ad}{bc} \in \mathbb{Q} \) since \( ad \) and \( bc \in \mathbb{Z}^+ \).

14. Let \( \mathbb{R}^+ \) denote the set of nonnegative reals. If \( g : \mathbb{R}^+ \to \mathbb{R} \) is defined by \( g(x) = \frac{1}{\sqrt{x}} + \sqrt{x} \), is \( g \) one-to-one?

No, \( g \) is not one-to-one. Note \( g(x) = g(y) \) \Rightarrow \( \frac{1}{\sqrt{x}} + \sqrt{x} = \frac{1}{\sqrt{y}} + \sqrt{y} \) \Rightarrow \( \frac{1}{x} + x = \frac{1}{y} + y \) \Rightarrow \( x - y = \frac{y - x}{xy} \) \Rightarrow \( xy = 1 \).

Hence, for example, \( g(1/2) = g(2) \).

15. Let \( A = \{1, 2, 3\} \) and \( B = \{1, 2\} \). How many functions are there from \( A \) to \( B \)?
There are 2 choices for \( f(1) \), for \( f(2) \), and for \( f(3) \). Hence, by the basic multiplicative counting principle, there are \( 2 \times 2 \times 2 \) choices for \( f \). In general, the number of functions equals \( c^d \) where \( c = \text{Card(codomain)} \) and \( d = \text{Card(domain)} \).

answer: \(_8_\)

16. Let \( A = \{1, 2, 3\} \) and \( B = \{1, 2, 3\} \). How many one-to-one functions are there from \( A \) to \( B \)?

Construct the one-to-one function sequentially. There are 3 choices for \( f(1) \), 2 choices for \( f(2) \), and 1 choice for \( f(3) \).

Hence, number of one-to-one functions = \( 3(2)(1) \).

In general, \# of one-to-one fcn = \( c(c-1)(c-2) \).

answer: \(_6_\)

17. Find a function \( f : D \to E \) such that \( f[f^{-1}[E]] \neq E \). [Keep it simple!]

Let \( f : \{0\} \to \{0, 1\} \) be defined by \( f(0) = 1 \). Here \( f[f^{-1}[E]] = \{1\} \neq \{0, 1\} = E \).

18. Let \( f : [0, 1] \to [0, 1] \) be defined by \( f(x) = x^3 \). Prove that \( f \) is a one-to-one function.

\[
f(x) = f(y) \\
\Rightarrow x^3 = y^3 \\
\Rightarrow x = y
\]

19. Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \). Let \( f : \mathbb{N} \to \mathbb{N} \) be defined by \( f(x) = 3x + 1 \). Prove that \( f \) is not surjective.

For example, let \( y = 2 \in \mathbb{N} \). There is no \( x \in \mathbb{N} \) such that \( f(x) = 2 \).

20. If \( A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4)\} \), find \( A \cap B \).
Since $A = \{1, 2\}$ and $B = \{2, 4\}$, $A \cap B = \{2\}$.

21. Let $A = \{1, 2\}$ and let $B = [1, 3]$. Graph $A \times B$.

22. Let $f = \{(x, x^2) : x \in [-2, 2]\}$. Prove that $f$ is not injective.

Since $f(-2) = 4 = f(2)$, $f$ is not one-to-one.

23. Let $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x, y) = xy - (x + y)$. Is the set of positive integers closed under $g$? (Justify your answer.)

No, since, for example, $g(1, 1) = 1 - 2 = -1 \notin \mathbb{Z}^+$, the set of positive integers.

24. Let $f$ be a function with domain $\{1, 2\}$ and codomain $\{1, 2, 3\}$. Explain why $f$ cannot be onto.

For a function with a finite domain, the cardinality of the range is no greater than the cardinality of the domain. Hence the range of $f$ has at most 2 elements and cannot equal $\{1, 2, 3\}$. Therefore $\text{range}(f) \neq \text{codomain}(f)$ and thus $f$ cannot be onto.

25. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function, then $f$ is injective.
Assume $f : \mathbb{R} \to \mathbb{R}$ is an increasing function. Let $x, y \in \mathbb{R}$ with $x \neq y$.
If $x > y$, then $f(x) > f(y)$ since $f$ is increasing.
If $y > x$, then $f(y) > f(x)$ since $f$ is increasing.
Therefore, $x \neq y$ implies $f(x) \neq f(y)$, and thus $f$ is injective.

26. Prove that $f[A \cap B] \subseteq f[A] \cap f[B]$. Complete the proof below.

Proof. Let $y \in f[A \cap B]$. Then there exists $x \in A \cap B$ such that $y = f(x)$.
Hence $x \in A$ with $y \in f(x)$, which implies $y \in f[A]$, and $x \in B$ with $y \in f(x)$,
which implies $y \in f[B]$. But $y \in f[A]$ and $y \in f[B]$ implies $y \in f[A] \cap f[B]$. \qed

27. Let $C = \{(x, y) : x^2 + y^2 = 1\}$ with $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Explain why $C$ is not a function.

The graph of $C$, a circle, fails the vertical line test. For example,
$(\sqrt{1/2}, -\sqrt{1/2}) \in C$ and $(\sqrt{1/2}, \sqrt{1/2}) \in C$.

28. Let $\Pi = \{\{a, b\}, \{c, d\}\}$ be a partition of $A = \{a, b, c, d\}$.
Find the equivalence relation $R$ induced by $\Pi$.

$R = \{(a, a), (a, b), (b, b), (b, a), (c, c), (c, d), (d, d), (d, c)\}$

29. Suppose $R$ is a transitive relation on the set $\{1, 2, 3, 4, 5\}$. If $(1, 3) \in R$, $(3, 5) \in R$, and $(5, 1) \in R$, what other ordered pairs must be elements of $R$?

$(1, 3) \in R$ and $(3, 5) \in R \Rightarrow (1, 5) \in R$
$(3, 5) \in R$ and $(5, 1) \in R \Rightarrow (3, 1) \in R$
$(5, 1) \in R$ and $(1, 3) \in R \Rightarrow (5, 3) \in R$
$(5, 1) \in R$ and $(1, 5) \in R \Rightarrow (5, 5) \in R$
$(1, 3) \in R$ and $(3, 1) \in R \Rightarrow (1, 1) \in R$
$(3, 1) \in R$ and $(1, 3) \in R \Rightarrow (3, 3) \in R$

answer: $(1, 1), (1, 5), (3, 1), (3, 3), (1, 5), (5, 5)$

30. Suppose $R$ is a reflexive relation on the set $\{1, 2, 3, 4, 5\}$.
List all ordered pairs that must be elements of \( R \).

**answer:** \((1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\)

31. Suppose \( R \) is a symmetric relation on the set \( \{1, 2, 3, 4, 5\} \).
   If \((1, 3) \in R\), what other ordered pair is an element of \( R \)?

   **answer:** \((3, 1)\)

32. Let \( R = \{(0,1), (0,2), (1,2), (2,1), (2,3), (1,3)\}\) be a relation on the set \( A = \{0, 1, 2, 3\} \).
   Prove that \( R \) **not** antisymmetric.

    **Since \((1, 2) \in R \) and \((2, 1) \in R, R \) is not antisymmetric.**

33. Define the relation \( R \) on \( \mathbb{N} = \{0, 1, 2, \ldots\} \) by \( xRy \) if and only if \( \frac{x-y}{3} \) is an integer.
   Find the equivalence class \([2]_R\).

    \([2]_R = \{2, 5, 8, 11, 14, 17, 20, \ldots\}\)

34. Let \( A = \{1, 2, 3, 4, 6, 8, 12\}\). Define a partial order by \( x \preceq y \) if and only if \( x \) divides \( y \).
   List all the maximal elements of poset \((A, \preceq)\).

     **8 and 12 are the only maximal elements. Element 8 is maximal since there is no \( y \in A, y \neq 8 \), such that 8 divides \( y \). Element 12 is maximal since there is no \( y \in A, y \neq 12 \), such that 12 divides \( y \). All the other elements of \( A \) divide 8 or 12.**

35. Let \( A = \{1, 2, 3, 4, 6, 8, 12\}\). Define a partial order by \( x \preceq y \) if and only if \( x \) divides \( y \).
   For poset \((A, \preceq)\), find the least upper bound of the set \( \{2, 3, 6\} \), if it exists.

     **6 is the least upper bound since 2 divides 6, 3 divides 6, and 6 divides all other upper bounds of \{2, 3, 6\}.**

36. Let \( A \subset \mathbb{R} \times \mathbb{R} \) defined by \( \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| + |y| \leq 2\} \).
Suppose $A$ is partitioned as shown below.
Find the equivalence relation $R$ induced by the partition.
For example, $(1, 1/2)R(−3/2, 1/4)$.

![Diagram](image)

---

37. Prove $f[A \cup B] = f[A] \cup f[B]$, that is, the image of a union equals the union of the images. Complete the proof below.

**Proof.** Let $y \in f[A \cup B]$. Then

$$y \in f[A \cup B] \iff \exists x \in A \cup B \text{ with } y = f(x)$$

$$\iff \exists x \in A \text{ with } y = f(x) \text{ or } \exists x \in B \text{ with } y = f(x)$$

$$\iff y \in f[A] \text{ or } y \in f[B].$$

---

38. Let $A = \{a, b, c\}$. Let $R$ be a relation on $A$ defined by $\{(a, a), (b, b), (c, c), (a, b), (a, c)\}$. Is $R$ a partial order of $A$? Justify your answer.

Yes, $R$ is a partial order on $A$ since

(i) $R$ is reflexive because $(a, a), (b, b)$ and $(c, c) \in R$,

(ii) $R$ is antisymmetric because $(b, a) \notin R$ and $(c, a) \notin R$, and

(iii) $R$ is transitive (trivially).

---

39. Let $A = \{0, 1, 2, 3, 4, 5\}$. Let $R = \{(x, y) \in A \times A : y - x = 3k \text{ for some integer } k\}$ be a relation on the set $A$. List the ordered pairs of $R$.

$$R = \{(0,0), (0,3), (3,0), (3,3), (1,1), (1,4), (4,1), (4,4), (2,2), (2,5), (5,2), (5,5)\}$$

---


Yes, because $R$ is induced by the partition of $A$ given by

$A = \{0, 3\} \cup \{1, 4\} \cup \{2, 5\}$. 

---

a. $R$ is defined by $(x, y)R(v, w)$ iff $|x| + |y| = |v| + |w|$

b. $R$ is defined by $(x, y)R(v, w)$ iff $|x| + |y| \leq |v| + |w|$

c. $R$ is defined by $(x, y)R(v, w)$ iff $|x| + |y| \geq |v| + |w|$

d. $R$ is defined by $(x, y)R(v, w)$ iff $|x| + |y| = |v| + |w|$