The Probability of a Tie in a Three Candidate Race

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Suppose an election has 3 candidates, $C_1$, $C_2$, and $C_3$, and $n$ voters cast ballots. If each of the candidates has an equally likely chance of getting voter's $j$ vote for $j = 1, \ldots, n$, what is the probability that $C_1$ and $C_2$ get an equal number of votes?

Let $N_i$ denote the number of votes for candidate $C_i$, $i = 1, 2, 3$. We want to find $P(N_1 = N_2)$.

We note that each $N_i$ has a binomial distribution with parameters $n$ and $p_i = 1/3$. Clearly, the $N_i$ are not independent random variables. Let $N_{12} = N_1 + N_2$.

Note that $N_{12}$ also has a binomial distribution with parameters $n$ and $p_{12} = 2/3$. To find $P(N_1 = N_2) = \sum_{k=0}^{[n/2]} P(N_1 = k, N_2 = k)$, consider the event $(N_1 = k, N_2 = k)$. This is equivalent to the compound event $(N_1 = k, N_{12} = 2k)$. But $P(N_1 = k, N_{12} = 2k) = P(N_{12} = 2k)P(N_1 = k|N_{12} = 2k)$. Note that $(N_1|N_{12} = 2k)$ is binomial$(2k, 1/2)$, and thus

$$P(N_1 = k|N_{12} = 2k) = \binom{2k}{k} \left(\frac{1}{2}\right)^{2k}.$$ 

We obtain

$$P(N_1 = N_2) = \sum_{k=0}^{[n/2]} P(N_1 = k, N_2 = k)$$

$$= \sum_{k=0}^{[n/2]} P(N_{12} = 2k)P(N_1 = k|N_{12} = 2k)$$

$$= \sum_{k=0}^{[n/2]} \binom{n}{2k} \left(\frac{2}{3}\right)^{2k} \left(\frac{1}{3}\right)^{n-2k} \binom{2k}{k} \left(\frac{1}{2}\right)^{2k}$$

$$= \left(\frac{1}{3}\right)^n \sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{2k}{k}$$

$$= \left(\frac{1}{3}\right)^n \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!k!}$$

$$= \left(\frac{1}{3}\right)^n t(n).$$

We note that the numbers $t(n) = \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!k!}$ form the sequence known as "central trinomial coefficients" (sequence A002426, On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences).