Gambler's Ruin Problem

Suppose a gambler starts a sequence of games with an initial amount of $i$ dollars. On each play she either wins $1$ with probability $p$ or loses $1$ with probability $q = 1 - p$. She plays until she's broke or has $N$ dollars (that is, wins $N - i$ dollars). Find $P_i$, the probability that she wins $N - i$ dollars.

**Solution.** Conditioning on the first play, we have $P_i = pP_{i+1} + qP_{i-1}$. Since we can write $P_i = pP_i + qP_i$, we obtain

$$p(P_{i+1} - P_i) = q(P_i - P_{i-1}),$$

which implies

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}).$$

Letting $D_k = P_k - P_{k-1}$, we have the recursive relationship

$$D_{i+1} = \frac{q}{p}D_i \text{ for } i = 1, 2, \ldots \text{ with } D_1 = P_1.$$

Hence

$$D_n = \left(\frac{q}{p}\right)^{n-1} P_1.$$

Now note that

$$P_i = P_1 + (P_2 - P_1) + \ldots + (P_i - P_{i-1})$$

$$= \sum_{k=1}^{i} D_k$$

$$= \sum_{k=1}^{i} \left(\frac{q}{p}\right)^{k-1} P_1$$

$$= P_1 \cdot \frac{1-(q/p)^{i+1}}{1-(q/p)}$$

provided $q \neq p$. If $q = p = \frac{1}{2}$, then $D_k = P_1$ and thus

$$P_i = \sum_{k=1}^{i} D_k = iP_1.$$

Furthermore, since $P_N = 1$, we find $P_1 = \frac{1-(q/p)}{1-(q/p)^N}$ when $p \neq q$ and $P_1 = \frac{1}{N}$ when $p = q$. Therefore,

$$P_i = \frac{1-(q/p)^i}{1-(q/p)^N} \text{ when } p \neq \frac{1}{2} \text{ and } (1)$$

$$P_i = \frac{i}{N} \text{ when } p = \frac{1}{2}. \quad (2)$$
Example. A gambler starts a sequence of games with an initial amount of \( i = 25 \) dollars. On each play she either wins $1 with probability \( p = .48 \) or loses $1 with probability \( q = 1 - p = .52 \). She plays until she's broke or has \( N = 30 \) dollars (that is, wins \( N - i = 5 \) dollars).

(i) Find \( P_{25} \), the probability that she wins \( 30 - 25 = 5 \) dollars.

Using the formula in (1) we have

\[
P_{25} = \frac{1 - (.52/48)^{25}}{1 - (.52/48)^{30}} = .63732
\]

Although the probability of winning here is greater than 1/2, why is this not a recommended betting scheme?

(ii) Find her expected winnings per sequence of games.

\[
E(W) = - $25 (.36268) + $5(.63732) = - $5.88
\]

The following table gives \( P_i \) for various \( N - i \) values when \( i = 25 \) and \( p = .48 \).

<table>
<thead>
<tr>
<th>( N - 25 )</th>
<th>( P_{25} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.91211</td>
</tr>
<tr>
<td>3</td>
<td>.76113</td>
</tr>
<tr>
<td>5</td>
<td>.63732</td>
</tr>
<tr>
<td>7</td>
<td>.53515</td>
</tr>
<tr>
<td>9</td>
<td>.45041</td>
</tr>
<tr>
<td>13</td>
<td>.32082</td>
</tr>
<tr>
<td>15</td>
<td>.27135</td>
</tr>
</tbody>
</table>

Expected winnings, \( E(W) \), per sequence when \( q > 1/2 \)

Assume \( q > 1/2 \). Then, the odds for losing a game, \( r = q/p > 1 \). We also assume that \( N > i \).

\[
E(W) = (N - i)P_i + (-i)(1 - P_i) = NP_i - iP_i - i + iP_i = NP_i - i
\]

We want to show that \( E(W) = NP_i - i < 0 \).

First, let \( N = i + k \) where \( k \in \mathcal{N} \). Then

\[
P_i = \frac{r^{i+k-1}}{r^{i+k} - 1}
\]

Hence \( E(W) < 0 \) iff \( NP_i - i < 0 \) iff \( (i + k)P_i - i < 0 \)
iff \( \frac{(i+k)}{k} P_i - \frac{i}{k} < 0 \)

iff \( \frac{i}{k} > \frac{(i+k)}{k} P_i \)

iff \( \frac{i}{k} (1 - P_i) > P_i \)

iff \( \frac{i}{k} > \frac{P_i^2}{(1-P_i)} \)

iff \( \frac{i}{k} > \frac{r^i-1}{r^{i+1} - r^i} \)

iff \( \frac{i}{k} > \frac{r^i-1}{r(r^i-1)} \)

iff \( \frac{i}{k} > \frac{r^i-1}{r(r^i-1)} \) (1).

We prove (1) using induction on \( i \).

For \( i = 1 \), we have \( \frac{1}{k} > \frac{1}{r+r^2+...+r^k} \) since \( r > 1 \).

But \( \frac{1}{r+r^2+...+r^k} = \frac{r-1}{r(r^k-1)} \) and hence \( \frac{1}{k} > \frac{r-1}{r(r^k-1)} \). (2)

Now assume \( \frac{i}{k} > \frac{r^i-1}{r(r^i-1)} \) for arbitrary positive integer \( i \).

Then
\[
\frac{i+1}{k} = \frac{i}{k} + \frac{1}{k}
\]
\[
> \frac{r^i-1}{r(r^i-1)} + \frac{1}{k} \quad \text{(by the induction hypothesis)}
\]
\[
> \frac{r^i-1}{r(r^i-1)} + \frac{r-1}{r(r^k-1)} \quad \text{(by (2))}
\]
\[
= \frac{r(r^i-1)+r^i(r-1)}{r^{i+1}(r^k-1)}
\]
\[
= \frac{r^{i+1}-(r-1)+r^i(r-1)}{r^{i+1}(r^k-1)}
\]
\[
= \frac{r^{i+1}+r(r-1)}{r^{i+1}(r^k-1)} \quad \text{since } r > 1
\]
\[
> \frac{r^{i+1}-1}{r^{i+1}(r^k-1)}
\]
\[
> \frac{r^{i+1}-1}{r^{i+1}(r^k-1)}
\]
Example. Suppose you start with two dollars and on each play you either win a dollar with probability .4 or lose a dollar with probability .6. You decide at the beginning that you will quit if you are ever ahead by $2 or if you go broke. Find the probability that you will end up winning $2.

Here the states will be the money at hand — 0, 1, 2, 3, or 4 dollars. Your initial state probability vector is \([0, 0, 1, 0, 0]\) since you start with 2 dollars. The transition probability matrix \(P\) is

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
.6 & 0 & .4 & 0 & 0 \\
0 & .6 & 0 & .4 & 0 \\
0 & 0 & .6 & 0 & .4 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

\[
P(\text{end up winning $2$}) = \frac{1-(q/p)^i}{1-(q/p)^N} = \frac{1-(.6/.4)^2}{1-(.6/.4)^4} = \frac{1-9/4}{1-81/16} = \frac{20}{65} = \frac{4}{13}.
\]

Alternatively, note

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0
\end{bmatrix}P = \begin{bmatrix}
0 & .6 & 0 & .4 & 0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0
\end{bmatrix}P^2 = \begin{bmatrix}
.36 & 0.48 & 0 & .16
\end{bmatrix}.
\]

After two plays you have probability .16 of winning $2 and probability .36 of being broke In fact,

\[
P(\text{end up winning $2$}) = \frac{16}{16+36} = \frac{4}{13}.
\]

Why? Because the possible winnings are double the initial stake. In general, if initial stake is \(i\) and one quits upon winning \(2i\) or going broke, then \(P(\text{end up winning $i$}) = P_i = \frac{p^i}{p^i+q^i} = \frac{p^i}{p^i+(1-p)^i}\).

For example if \(p = .49\), then \(P_i = .49^i/(.49^i + .51^i)\).

<table>
<thead>
<tr>
<th>(i)</th>
<th>(P_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.49</td>
</tr>
<tr>
<td>5</td>
<td>.45016</td>
</tr>
<tr>
<td>10</td>
<td>.4013</td>
</tr>
<tr>
<td>15</td>
<td>.35433</td>
</tr>
<tr>
<td>20</td>
<td>.31</td>
</tr>
<tr>
<td>25</td>
<td>.26892</td>
</tr>
<tr>
<td>27</td>
<td>.25348</td>
</tr>
</tbody>
</table>
Example. A gambler starts a sequence of games with an initial amount of $i = 25$ dollars. On each play she either wins $1$ with probability $p = .48$ or loses $1$ with probability $q = 1 - p = .52$. She plays until she's broke or has $N = 40$ dollars (that is, wins $N - i = 15$ dollars).

(i) Find $P_{25}$, the probability that she wins $40 - 25 = 15$ dollars.

Using the formula in (1) we have

$$P_{25} = \frac{1 - (.52/.48)^{25}}{1 - (.52/.48)^{40}} = .2713506671$$

(ii) What is her expected winnings per sequence of games?

$$E(W) = (1 - .2713506671)(-25) + (.2713506671)(15) \approx -14.15$$