1. If a fair coin is successively flipped, find the probability that a head first appears on the sixth trial.

Let \( X \) denote the trial on which the first head occurs. Then

\[
P(X = 6) = P(\text{event of coin sequence } T, T, T, T, T, H)
\]

\[
= \left( \frac{1}{2} \right)^6
\]

\[
= \frac{1}{64}.
\]

2. An individual claims to have extrasensory perception (ESP). As a test, a fair coin is flipped ten times, and she is asked to predict in advance the outcome. Our individual gets 7 out of 10 correct. What is the probability that she would have done at least this well if she had no ESP?

Let \( X \) denote the number of correct predictions. Assuming she has no ESP, \( X \) has a Binomial\( (n = 10, p = 1/2) \) distribution. Hence we have

\[
P(X \geq 7) = \sum_{k=7}^{10} \binom{10}{k} \left( \frac{1}{2} \right)^{10} = \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}
\]

\[
= \frac{120 + 45 + 10 + 1}{1024} = .171875
\]

Using \( TI - 83 : 1 - \text{binomcdf}(10, .5, 6) \)

3. Let \( X \) be a random variable with probability density given by

\[
f(x) = c(4x - 2x^2) I_{(0,2)}(x).
\]

Find the necessary value for \( c \) and then calculate \( E\{X\} \).

Since \( \int_0^2 4x - 2x^2 \, dx = \left( 2x^2 - \frac{2}{3}x^3 \right)_0^2 = 8 - 16/3 = 8/3 \), \( c \) equals \( 3/8 \) in order that \( \int f(x) \, dx = 1 \). Furthermore,

\[
E(X) = \frac{3}{8} \int_0^2 x(4x - 2x^2) \, dx = \frac{3}{8} \int_0^2 4x^2 - 2x^3 \, dx
\]

\[
= \frac{3}{8} \left( \frac{4}{3}x^3 - \frac{1}{2}x^4 \right)_0^2 = \frac{3}{8} (32/3 - 8) = 1.
\]

\textit{Note.} The graph of the pdf above is a parabolic curve that is symmetric about the line \( x = 1 \).
4. Let $X$ be a discrete random variable with support $\{0, 1, 2, \ldots\}$. If $F$ denotes the cumulative distribution function of $X$, show that $E\{X\} = \sum_{x=0}^{\infty} [1 - F(x)]$.

\[
E\{X\} = \sum_{x=0}^{\infty} x f(x) \quad \text{[where } f \text{ is the probability mass function for } X]\]

\[
= 0 f(0) + 1 f(1) + 2 f(2) + 3 f(3) + 4 f(4) + \ldots
\]

\[
= [f(1) + f(2) + f(3) + f(4) + \ldots] + [f(2) + f(3) + f(4) + \ldots] + [f(3) + f(4) + \ldots] + f(4) + \ldots
\]

\[
= [1 - F(0)] + [1 - F(1)] + [1 - F(2)] + [1 - F(3)] + \ldots
\]

\[
= \sum_{x=0}^{\infty} [1 - F(x)].
\]

5. Suppose that $X$ is a random variable with mean 10 and variance 15. What can we say about $X$?

By Chebyshev’s inequality, $P\{\mu - k\sigma < X < \mu + k\sigma\} \geq 1 - \frac{1}{k^2}$ for $k > 1$.

Since $5 = 10 - k\sqrt{15}$ and $15 = 10 + k\sqrt{15}$ implies $k = 5/\sqrt{15}$ implies $k = 5/\sqrt{15}$. Hence

\[
P\{5 < X < 15\} \geq 1 - \left(\frac{\sqrt{15}}{5}\right)^2 = 1 - \frac{15}{25} = \frac{10}{25} = .40.
\]

6. Let $X_1, X_2, \ldots, X_{10}$ be independent Poisson random variables with mean 1.

(i) Use the Markov inequality to get a bound on $P\{\sum_{i=1}^{10} X_i \geq 12\}$.

Markov’s inequality states that, whenever $Y$ is a nonnegative random variable, $E(Y) \geq cP(Y \geq c)$ for every $c > 0$. Let $Y = \sum_{i=1}^{10} X_i$ and note that $E\{\sum_{i=1}^{10} X_i\} = 10$. Therefore

\[
P\{\sum_{i=1}^{10} X_i \geq 12\} \leq \frac{1}{12} E\{\sum_{i=1}^{10} X_i\} = \frac{1}{12} \cdot 10 = \frac{5}{6}.
\]
(ii) Use the central limit theorem to approximate \( P\left\{ \sum_{i=1}^{10} X_i \geq 12 \right\} \).

Note that \( \text{Var}\{Y\} = \text{Var}\{\sum_{i=1}^{10} X_i\} = 10 \). Hence

\[
P\left\{ \sum_{i=1}^{10} X_i \geq 12 \right\} = P\{Y \geq 11.5\}
\approx P\{Z \geq \frac{11.5-10}{\sqrt{10}}\}
= P\{Z \geq 0.474341649\}
= .3176
\]

7. Let \( X \) denote the number of red balls selected when 10 balls are chosen at random from an urn containing 20 red balls and 30 green balls. Find \( E\{X\} \).

Here \( X \) has a hypergeometric distribution. Thus

\[
E\{X\} = (\text{number of balls chosen})(\text{proportion of balls that are red})
= 10 \cdot \frac{20}{50} = 4.
\]

8. Calculate the moment generating function for random variable \( X \) whose probability mass function is given by \( f(x) = \frac{1}{3} \left( \frac{2}{3} \right)^x I_{\{0,1,\ldots\}}(x) \).

\[
M_X(t) = E\{e^{tX}\}
= \sum_{x=0}^{\infty} e^{tx} \frac{1}{3} \left( \frac{2}{3} \right)^x
= \frac{1}{3} \sum_{x=0}^{\infty} \left( \frac{2}{3} e^t \right)^x = \frac{1}{3} \frac{1}{1-\frac{2}{3}e^t} = \frac{1}{3-2e^t} \quad \text{for} \quad |\frac{2}{3}e^t| < 1 \text{ or } t < \ln \frac{3}{2}.
\]

9. In deciding upon the appropriate premium to charge, insurance companies sometimes use the exponential principle, defined as follows. With \( X \) as the random amount that it will have to pay in claims, the premium charged by the insurance company is

\[
P = \frac{1}{a} \ln(E[e^{aX}])
\]

where \( a \) is some specified positive constant. Find \( P \) when \( X \) is an exponential random variable with p.m.f. \( f(x) = \frac{1}{10} e^{-x/10} I_{(0,\infty)}(x) \) and \( a = 1/20 \).

Here \( E[e^{aX}] = \int_{0}^{\infty} e^{x/20} \frac{1}{10} e^{-x/10} dx = 2 \) and so \( P = 20 \ln(2) \).