Derivations for Quantities Related to the Least Squares Slope Estimator

Let \( Q = \sum_{i=1}^{n} (Y_i - E(Y_i))^2 = \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_i)^2 \). Find values \( b_0 \) and \( b_1 \) for \( \beta_0 \) and \( \beta_1 \) that minimize \( Q \). Using calculus (partial derivatives), we find

\[
b_1 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} = \frac{SSXY}{SSX}
\]

and

\[
b_0 = \bar{Y} - b_1 \bar{X}
\]

1. Note \( SSXX = \sum_{i=1}^{n} (X_i - \bar{X})^2 \)

\[
= \sum_{i=1}^{n} (X_i - \bar{X})X_i - \bar{X} \sum_{i=1}^{n} (X_i - \bar{X})
\]

\[
= \sum_{i=1}^{n} (X_i - \bar{X})X_i
\]

2. Note \( SSXY = \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}) \)

\[
= \sum_{i=1}^{n} (X_i - \bar{X})Y_i - \bar{Y} \sum_{i=1}^{n} (X_i - \bar{X})
\]

\[
= \sum_{i=1}^{n} (X_i - \bar{X})Y_i
\]

3. Show \( b_1 = \sum_{i=1}^{n} k_i Y_i \) where \( k_i = \frac{X_i - \bar{X}}{\sum_{i=1}^{n} (X_i - \bar{X})^2} = \frac{X_i - \bar{X}}{SSXX} \).

\[
b_1 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n} (X_i - \bar{X})^2} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})Y_i}{\sum_{i=1}^{n} (X_i - \bar{X})^2} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})Y_i}{SSXX} = \frac{n}{SSXX} \sum_{i=1}^{n} \frac{(X_i - \bar{X})}{SSXX} Y_i = \sum_{i=1}^{n} k_i Y_i.
\]
4. Show \[ \sum_{k=1}^{n} k_i = 0 \text{ and } \sum_{k=1}^{n} k_i^2 = \frac{1}{SSXX}. \]

\[ \sum_{k=1}^{n} k_i = \sum_{i=1}^{n} \frac{X_i - \overline{X}}{SSXX} = \frac{1}{SSXX} \sum_{i=1}^{n} (X_i - \overline{X}) = \frac{1}{SSXX} (0) = 0 \]

and

\[ \sum_{k=1}^{n} k_i^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{(SSXX)^2} = \frac{1}{(SSXX)^2} \sum_{i=1}^{n} (X_i - \overline{X})^2 = \frac{1}{(SSXX)^2} SSXX = \frac{1}{SSXX}. \]

5. Show \[ \sum_{k=1}^{n} k_i X_i = 1. \]

\[ \sum_{k=1}^{n} k_i X_i = \sum_{k=1}^{n} \frac{X_i - \overline{X}}{SSXX} X_i = \frac{1}{SSXX} \sum_{i=1}^{n} (X_i - \overline{X}) X_i = \frac{SSXX}{SSXX} = 1 \]

6. Show \[ E(b_1) = \beta_1. \] Since \[ E(Y_i) = \beta_0 + \beta_1 X_i, \] we have

\[ E(b_1) = E \left( \sum_{i=1}^{n} k_i Y_i \right) \]

\[ = \sum_{i=1}^{n} k_i E(Y_i) \]

\[ = \sum_{i=1}^{n} k_i (\beta_0 + \beta_1 X_i) \]

\[ = \beta_0 \sum_{i=1}^{n} k_i + \beta_1 \sum_{i=1}^{n} k_i X_i, \]

\[ = \beta_1 \]

7. Show \[ Var(b_1) = \frac{\sigma^2}{SSXX} \] where \[ \sigma^2 = Var(\epsilon_i) \] for all \[ i. \]

\[ Var(b_1) = Var( \sum_{k=1}^{n} k_i Y_i ) = \sum_{k=1}^{n} k_i^2 Var(Y_i) \]

\[ = \sum_{k=1}^{n} k_i^2 Var(\epsilon_i) = \sigma^2 \sum_{k=1}^{n} k_i^2 \]

\[ = \frac{\sigma^2}{SSXX} \]