A Short Note of Increasing Acyclic Functions

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A function $f$ with real domain $D$ and real codomain $C$, with $D \subseteq C$, is called 
acyclic if $f(B) \neq B$ for every $B \subseteq D$. Equivalently, $f$ is acyclic if, for every $x \in D$, 
there exists integer $k$ such that $f^k(x) \in C - D$. In other words, an acyclic function 
“eventually sends” (under successive composition) every element of the domain to the 
the complement of the domain in $C$.

Example. Let $f: \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8\}$ be defined by 
f = \{(1, 3), (2, 8), (3, 2), (4, 6), (5, 4)\}. Let $E = C - D = \{6, 7, 8\}$. 
Note that 
f^3(1) = 8 \in E, 
f(2) = 8 \in E, 
f^2(3) = 8 \in E 
f(4) = 6 \in E, 
f^2(5) = 6 \in E.

Therefore, $f$ is acyclic.

Recall that an increasing function $f$ satisfies $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Example. Let $f: \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8\}$ be defined by 
f = \{(1, 2), (2, 4), (3, 6), (4, 7), (5, 8)\}. Function $f$ is acyclic and increasing.

Theorem. Let $\mathcal{F}$ denote the set of increasing acyclic functions with a finite codomain. If 
the domain has cardinality $n$ and the codomain has cardinality $n + m$, then the 
cardinality of $\mathcal{F}$ is given by $c(n, m) = \binom{n+m-1}{n}$.

Proof. Without loss of generality, let $f: \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n + m\}$. If $f$ is acyclic, 
then $f(1) > 1$. If $f$ is increasing, then $f(1) < f(2) < \ldots < f(n) \leq n + m$. Therefore, 
if $f$ is increasing and acyclic, $2 \leq f(1) < f(2) < \ldots < f(n) \leq n + m$. To construct 
such a function $f$, we need only chose $n$ elements from $\{2, 3, \ldots, n + m\}$ to be $f(1)$, 
f(2), \ldots, $f(n)$, respectively. There are exactly $\binom{n+m-1}{n}$ ways to do this. 
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