A Short Note of Increasing Acyclic Functions

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A function f with real domain D and real codomain C, with $D \subseteq C$, is called *acyclic* if $f(B) \neq B$ for every $B \subseteq D$. Equivalently, f is acyclic if, for every $x \in D$, there exists integer k such that $f^k(x) \in C - D$. In other words, an acyclic function "eventually sends" (under successive composition) every element of the domain to the the complement of the domain in C.

Example. Let f: $\{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8\}$ be defined by $f = \{(1, 3), (2, 8), (3, 2), (4, 6), (5, 4)\}$. Let $E = C - D = \{6, 7, 8\}$.
Note that

$$f^3(1) = 8 \in E,$$
$$f(2) = 8 \in E,$$
$$f^2(3) = 8 \in E$$
$$f(4) = 6 \in E,$$
$$f^2(5) = 6 \in E.$$

Therefore, f is acyclic.

Recall that an increasing function f satisfies $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Example. Let f: $\{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8\}$ be defined by $f = \{(1, 2), (2, 4), (3, 6), (4, 7), (5, 8)\}$. Function f is acyclic and increasing.

Theorem. Let $\mathcal{F}$ denote the set of increasing acyclic functions with a finite codomain. If the domain has cardinality n and the codomain has cardinality $n + m$, then the cardinality of $\mathcal{F}$ is given by $c(n, m) = \binom{n+m-1}{n}$.

Proof. Without loss of generality, let $f : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., n + m\}$. If f is acyclic, then $f(1) > 1$. If f is increasing, then $f(1) < f(2) < ... < f(n) \leq n + m$. Therefore, if f is increasing and acyclic, $2 \leq f(1) < f(2) < ... < f(n) \leq n + m$. To construct such a function f, we need only chose n elements from $\{2, 3, ..., n + m\}$ to be $f(1)$, $f(2), ..., f(n)$, respectively. There are exactly $\binom{n+m-1}{n}$ ways to do this. $\square$