Toy Stories and Combinatorial Identities

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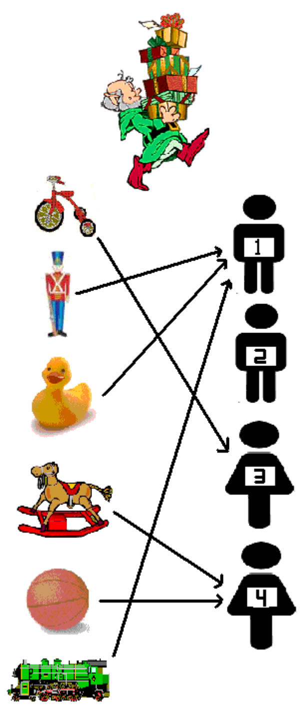
I. Toy Stories

Suppose an elf has n different toys that need to be distributed among x boys and y girls with no restrictions on how many toys a particular child receives. Then $(x+y)^n$ represents the number of possible ways to distribute the toys since there are $x+y$ ways to pick the child who gets toy 1, $x+y$ ways to pick the child who gets toy 2, etc. On the other hand, if we count the distributions by conditioning on the number of toys the boys receive and the number of toys the girls receive, we generate an expansion of $(x+y)^n$. Since there are $\binom{n}{k}$ ways to choose k toys for the boys and $n-k$ toys for the girls, x^k ways to distribute the k different toys among the x boys, and y^{n-k} ways to distribute remaining $n-k$ different toys among the y girls, we obtain

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad (1)$$

the familiar binomial theorem.

Example 1. For $n=6$, $x=2$, and $y=2$, there are 4^6 possible toy distributions. Note that identity (1) implies $4^6 = \sum_{k=0}^{6} \binom{6}{k} 2^k 2^{n-k}$. One particular distribution is shown below.



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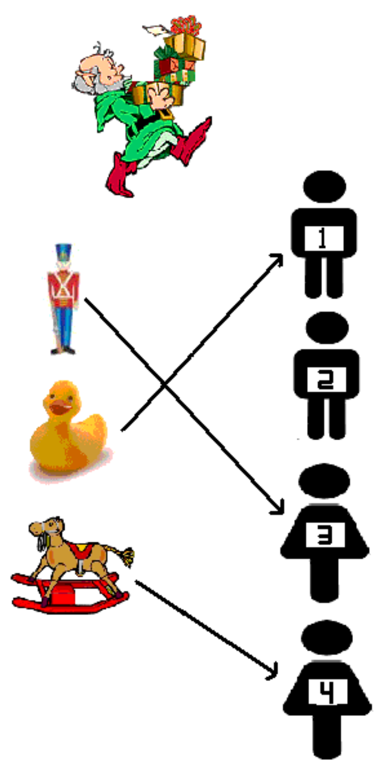
Other similar “toy story” identities are possible, and we develop them below.

How many distributions are possible if the n distinct toys are to be distributed among the x boys and the y girls so that every child gets no more than one toy?. If we let $P(r,s)$ denote the number of permutations of r items taken s at a time, that is $P(r,s) = \frac{r!}{(r-s)!}$, then $P(\ x+y, n)$ represents the number of possible distributions since there are $(x+y)$ ways to pick the child that gets toy 1, $(x+y-1)$ ways to pick the child that gets toy 2, etc. On the other hand, if we count these distributions by focusing on the number of toys the boys get and the number of toys the girls get, we arrive at $\sum_{k=0}^{n} \binom{n}{k} P(x,k)P(y,n-k)$. Hence we obtain the identity

$$P(x+y,n) = \sum_{k=0}^{n} \binom{n}{k} P(x,k)P(y,n-k), \qquad (2)$$

whose form mimics that in (1). We note that (2) is also known as Vandermonde's formula and proven in Berge [1] using a Taylor formula and the forward difference operator.

Example 2. For $n=3$, $x=2$, and $y=2$, there are $P(4,3) = 4(3)(2) = 24$ possible toy distributions. Note that (2) implies $P(4,3) = \sum_{k=0}^{3} \binom{3}{k} P(2,k)P(2,3-k)$. One particular distribution is shown below.



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In a similar fashion, if we require that every boy and every girl gets at least one toy, to find the number of possible toy distributions we must count the number of surjections from a set of size n to a set of size $x+y$. Letting $T(n,\, x+y)$ denote the number of surjections and then counting again by conditioning on the number of toys the boys receive, we obtain

$$T(n, x+y) = \sum_{k=0}^{n} \binom{n}{k} T(k, x) T(n-k, y) \tag{3}$$

where $T(r, s)$ is given by

$$T(r, s) = \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} i^{r} \tag{4}$$

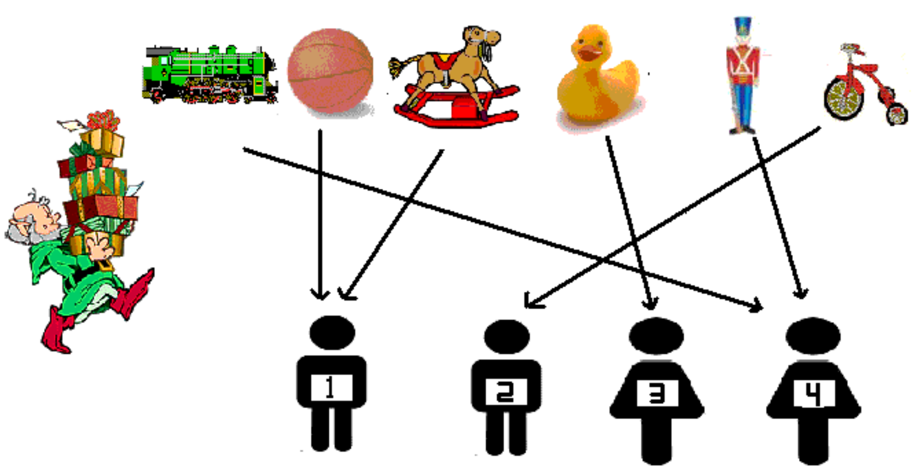
$$= s!\, S_2(r, s)$$

and where $S_2(r, s)$ denotes a Stirling number of the second kind. Therefore, identity (3) can be written as

$$(x+y)! S_2(n, x+y) = \sum_{k=0}^{n} \binom{n}{k} x! S_2(k, x) y! S_2(n-k, y). \tag{3a}$$

Example 3. For $n = 6$, $x = 2$, and $y = 2$, there are $T(6,4)$ ways to distribute 6 different toys among 2 boys and 2 girls so that each child receives at least one toy. Note that (3) implies $T(6,4) = \sum_{k=0}^{6} \binom{6}{k} T(k,2) T(3-k,2)$ and (4) implies $T(6,4) = \sum_{i=0}^{4} (-1)^{4-i} \binom{4}{i} i^6 = 0^6 - \binom{4}{1} 1^6 + \binom{4}{2} 2^6 - \binom{4}{3} 3^6 + \binom{4}{4} 4^6 = 1560.$

Below is one such distribution.



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II. Characterizing Function Properties for the Toy Story Identities

The toy stories above illustrate combinatorial properties of certain sets of finite functions. We will formalize these properties and present a theorem that generalizes the binomial identities in (1), (2), and (3). We first revisit the initial first toy story, but now couched in terms of functions and partitions.

For a finite set S, let $|S|$ denote the cardinality of S. If x and y are positive integers, then $(x+y)^n$ denotes the number of functions $f{:}D{\to}C$ where $|D| = n$ and $|C| = x+y$. Without loss of generality let the domain $D = \{1, \ldots, n\}$, let the codomain $C = \{1, \ldots, x+y)$, and let $F = \{f : D{\to}C\}$. Let $f^{-1}\{A\}$ denote the preimage of A under f. If we partition C into $C_1 = \{1, \ldots, x\}$ and $C_2 = \{x+1, \ldots, x+y\}$, each function $f \in F$ is partitioned into $f_1 : f^{-1}\{C_1\}{\to}C_1$ and $f^{-1}\{C_2\}{\to}C_2$ such that $f = f_1 \cup f_2$. Furthermore, the partition of the codomain C also induces a partition of F into $n+1$ distinct subsets of the form

$$F_k = \{f = f_1 \cup f_2 \colon |f^{-1}\{C_1\}| = k \text{ and } |f^{-1}\{C_2\}| = n-k\}$$

such that

$$F = \bigcup_{k=0}^{n} F_k.$$

There are $\binom{n}{k}$ ways to choose the k elements of $f^{-1}\{C_1\}$ and, by default, to choose the elements of $f^{-1}\{C_2\}$. Furthermore, there are x^k ways to map k elements to a codomain of size x, and there are y^{n-k} ways to map $n-k$ elements to a codomain of size y. Hence we obtain

$$\begin{aligned}(x+y)^n &= |F| \\ &= \left|\bigcup_{k=0}^{n} F_k\right| \\ &= \sum_{k=0}^{n} |F_k| \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.\end{aligned}$$

We will extend our counting to sets of finite integer functions that have special properties. With appropriate restrictions on finite functions, we will show that the cardinality of the sets follow a binomial identity that mimics the binomial theorem. We first provide several necessary definitions (of particular function types and of particular properties these function may possess).

Definition. For finite sets of integers D and C, a function $f : D{\to}C$ will be called a *finite integer function*. In the case where $D = \emptyset$, we define $f : \emptyset{\to}C$, for any C, as the *null function*. On the other hand, when D is nonempty and $C = \emptyset$, we let $f : D{\to}\emptyset$ be undefined.

Example 4. The function $f : \{1,2,3\} \to \{10,11,12\}$ defined by

x	1	2	3
$f(x)$	12	10	10

is a finite integer function since both the domain $\{1,2,3\}$ and the codomain $\{10,11,12\}$ are finite sets of integers.

Definition. Two functions are said to be *completely disjoint* if their domains are disjoint and their codomains are disjont. Formally, functions $g : D_1 \to C_1$ and $h : D_2 \to C_2$ are called *completely-disjoint* functions if $D_1 \cap D_2 = \emptyset$ and $C_1 \cap C_2 = \emptyset$.

Example 5. The functions $g : \{1,3,4\} \to \{3,5,7,9\}$ and $h : \{2,5,6,8\} \to \{1,2,4,6\}$ are completely-disjoint functions since $\{1,3,4\} \cap \{2,5,6,8\} = \emptyset$ and $\{3,5,7,9\} \cap \{1,2,4,6\} = \emptyset$.

Definition. The *union* of two completely-disjoint functions $g : D_1 \to C_1$ and $h : D_2 \to C_2$ is a function $f : (D_1 \cup D_2) \to (C_1 \cup C_2)$ defined by

$$f(x) = \begin{cases} g(x) \text{ if } x \in D_1 \\ h(x) \text{ if } x \in D_2 \end{cases} .$$

Furthermore, we say that g and h *partition* f.

Example 6. Let $g : \{1,3,4\} \to \{3,5,7,9\}$ and $h : \{2,5,6,8\} \to \{1,2,4,6\}$. Then $f = g \cup h$ is defined by

x	1	2	3	4	5	6	8
$f(x)$	$g(1)$	$h(2)$	$g(3)$	$g(4)$	$h(5)$	$h(6)$	$h(8)$

Definition. Let F denote a set of finite integer functions that satisfies some property p. Formally, for nonnegative integers n and r, let $F = \{f : D \to C$ such that $|D| = n$, $|C| = r$, and f satisfies some property $p\}$. The property p satisfied by all functions in F is called a *consistent property* if (i) the cardinality c of F is a function that depends only on n and r, that is, $|F| = c = c(n,r)$; (ii) whenever two completely-disjoint functions have property p then their union also has property p; (iii) whenever a function f has property p, then any two completely-disjoint functions that partition f also have property p.

Example 7. The "intrinsic" property, that is, the property satisfied by all functions, is a consistent property. Let $F = \{f : D \to C$ such that $|D| = n$ and $|C| = r\}$. Note that the cardinality of F is r^n, which is a function of only r and n. Also, the union of any two completely-disjoint functions is also a function. Furthermore, any two-part partitioning of a function results in two functions.

Example 8. Injectiveness is a consistent property. If $F_I = \{f : D \rightarrow C \mid f$ is injective, $|D| = n$, and $|C| = r\}$, then the cardinality c of F_I is given by $c(n, r) = r!/(r - n)!$, which depends only on n and r. To show condition (ii) of a consistent function, assume g and h are two completely-disjoint injective functions and $f = g \cup h$. If $f(x) = f(y)$, then [since range$(g) \cap$ range(h) because g and h are completely-disjoint] either $g(x) = g(y)$ or $h(x) = h(y)$. In either case, since both g and h are injective, we have $x = y$. To show condition (iii) of the definition, let $f{:}D \rightarrow C$ be an injective function and let $f = g \cup h$ where g and h are any two completely-disjoint functions that partition f. If $g(x) = g(y)$, then $f(x) = f(y)$ since $f = g \cup h$. Hence $x = y$ (since f is injective) and thus g is injective. Also, if $h(x) = h(y)$, then $f(x) = f(y)$ since $f = g \cup h$. Hence $x = y$ (since f is injective) and thus h is injective.

Example 9. Surjectiveness is a consistent property, and monotonicity is not a consistent property.

The following theorem generalizes the identities contained in (1), (2), and (3).

Theorem. Let $F = \{f : D \rightarrow C \mid f$ is a finite integer function with a consistent property $p\}$. Let $c(n, r)$ denote the cardinality of F where n is the cardinality of D and r is the cardinality of C. Then

$$c(n, r) = \sum_{k=0}^{n} \binom{n}{k} c(k, r_1) c(n - k, r_2)$$

where r_1 and r_2 are nonnegative integers satisfying $r_1 + r_2 = r$.

Proof. For $i = 1, .., \binom{n}{k}$, let D_{ki} be a size-k subset of D, let C_1 be a fixed size r_1-subset of C, and let $C_2 = C \backslash C_1$. Let G_{ki} denote the set of functions defined by

$$G_{ki} = \{g : D_{ki} \rightarrow C_1 \mid g \text{ has consistent property } p\},$$

and let H_{ki} denote the set of functions

$$H_{ki} = \{h : D \backslash D_{ki} \rightarrow C_2 \mid h \text{ has consistent property } p\}.$$

For every function pair $(g, h) \in G_{ki} \times H_{ki}$, we have $g \cup h \in F$ since $g \cup f$ has domain D, codomain C, and consistent property p. The respective cardinalities of G_{ki} and of H_{ki} are $c(k, r_1)$and $c(n - k, r - r_1)$ since the property p is a consistent property. Therefore, for each i, the cardinality of $G_{ki} \times H_{ki}$ is $c(k, r_1) \cdot c(n - k, r - r_1)$.

Now let $F_k = \bigcup_i \{(g \cup h) : (g, h) \in G_{ki} \times H_{ki}\}$ where $i = 1, ..., \binom{n}{k}$. Note that for $j \neq k$ the sets F_j and F_k are disjoint. Also, since each function of F_k is a function from $D \rightarrow C$ with the consistent property p, each function of F_k is a function of F. On the other hand, for each function $f \in F$, we have $f^{-1}(C_1) = D_{ki}$ and $f^{-1}(C_2) = D \backslash D_{ki}$ for some $k \in \{0, 1, ..., n\}$] and for some $i \in \{1, ..., \binom{n}{k}\}$, which implies each f in F is also in F_k for some k. Therefore $F = \bigcup_{k=0}^{n} F_k$ and hence the cardinality of F is given by

$$
\begin{aligned}
|F| &= \sum_{k=0}^{n} |F_k| \\
&= \sum_{k=0}^{n} |\bigcup_i \{(g \cup h) : (g, h) \in G_{ki} \times H_{ki}| \\
&= \sum_{k=0}^{n} \sum_{i=1}^{\binom{n}{k}} |G_{ki} \times H_{ki}| \\
&= \sum_{k=0}^{n} \binom{n}{k} c(k, r_1) \cdot c(n-k, r-r_1).
\end{aligned}
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Reference

Berge, C., *Principles of Combinatorics*, Academic Press, New York, 1971.