BINOID POLYNOMIALS

Dennis Walsh
Middle Tennessee State University

Let \( P_n(x) \) represent a polynomial in \( x \) of degree \( n \), where \( n \) is a nonnegative
integer, \( x \) is a real number, and, by convention, \( P_0(x) = 1 \) for all \( x \). Consider a family (or
sequence) of polynomials \( \{1, P_1(x), P_2(x), P_3(x), \ldots \} \) that satisfies the following
binomial property for all real numbers \( x \) and \( y \):

\[
P_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y)
\]

(1)

for \( n \in \{0, 1, 2, \ldots \} \). We shall call a family of polynomials that satisfies identity (1) a
\textit{binoid} polynomial family. An arbitrary member, \( P_k(x) \), of a binoid polynomial
family will be called a \textit{binoid} polynomial. [The term \textit{binoid} comes from \textit{binomial}
identity.]

A method to generate binoid polynomial families is given in the following
theorem. For a function \( f(t) \), let \( D^k f(a) \) denote the \( k \)-th derivative of \( f \) evaluated at
\( t = a \), and let \( D^0 f(a) \) denote \( f(a) \).

\textbf{Theorem 1.} Let \( f \) be a real-valued function with the following properties:

(i) \( f(0) = 1 \);
(ii) \( f'(0) \neq 0 \);
(iii) \( D^n f(0) \) exists for any positive integer \( n \).

If function \( g_x \) is defined by \( g_x(t) = [f(t)]^x \) for all real \( x \), then \( D^n g_x(0) \) is a binoid
polynomial in \( x \) of degree \( n \) for \( n \in \{0, 1, \ldots \} \). In other words, \( \{D^n g_x(0)\}_{n=0}^{\infty} \) is a
binoid polynomial family.

\textbf{Proof.} (i) We first show that \( D^n g_x(0) \) is a polynomial in \( x \) of degree \( n \). Note that
\( D^0 g_x(0) = g_x(0) = 1 \), a polynomial of degree 0. Next we use induction to show, for
positive integer \( n \), that

\[
D^n g_x(t) = \sum_{k=1}^{n} x^{(k)} f(t)^{x-k} c_{n,k}(t)
\]

(2)

where \( \{c_{n,k}(t)\}_{k=1}^{n} \) is some sequence of functions of \( t \) with \( c_{n,n}(t) = [f'(t)]^n \). For
\( n = 1 \), the chain rule gives us \( D g_x(t) = D [f(t)]^x = x \cdot f(t)^{x-1} \cdot f'(t) \), which is of the
form in (2). Now assume (2) holds for arbitrary \( n > 1 \). Then
\[ D^{n+1}g_x(t) = D(D^n g_x(t)) \]
\[ = D\left(\sum_{k=1}^{n} x^{(k)} f(t)^{x-k} c_{n,k}(t)\right) \]
\[ = \sum_{k=1}^{n} (x - k) x^{(k)} f(t)^{x-k-1} f'(t) c_{n,k}(t) + \sum_{k=1}^{n} x^{(k)} f(t)^{x-k} c'_{n,k}(t). \]

Now re-index the first summation using \( j = k + 1 \) to get
\[ D^{n+1}g_x(t) = \sum_{j=2}^{n+1} (x - j + 1) x^{(j-1)} f(t)^{x-j} f'(t) c_{n,j-1}(t) + \sum_{k=1}^{n} x^{(k)} f(t)^{x-k} c'_{n,k}(t). \]

But since \((x - j + 1)x^{(j-1)} = x(x - 1) \cdots (x - j + 2)(x - j + 1) = x^{(j)}\), we obtain
\[ D^{n+1}g_x(t) = \sum_{j=2}^{n+1} x^{(j)} f(t)^{x-j} f'(t) c_{n,j-1}(t) + \sum_{k=1}^{n} x^{(k)} f(t)^{x-k} [f'(t)c_{n,k-1}(t) + c'_{n,k}(t)] \]
\[ + xf(t)^{x-1}c'_{n,1}(t). \]

If we define \( c_{n,0}(t) \) and \( c'_{n,0}(t) \) to be identically 0, we have
\[ D^{n+1}g_x(t) = \sum_{k=1}^{n+1} x^{(k)} f(t)^{x-k} [f'(t)c_{n,k-1}(t) + c'_{n,k}(t)]. \]

Let the sequence of functions \( \{d_{n+1,k}(t)\}_{k=1}^{n+1} \) be defined by \( d_{n+1,k}(t) = f'(t)c_{n,k-1}(t) + c'_{n,k}(t) \). Therefore
\[ D^{n+1}g_x(t) = \sum_{k=1}^{n+1} x^{(k)} f(t)^{x-k} d_{n+1,k}(t) \]
with \( d_{n+1,n+1}(t) = f'(t)c_{n,n}(t) + c'_{n,n}(t) = f'(t)[f'(t)]^n = [f'(t)]^{n+1} \), completing the induction proof.

Since (2) holds, we now have
\[ D^n g_x(0) = \sum_{k=1}^{n} x^{(k)} f(0)^{x-k} c_{n,k}(0) \]
\[ = c_{n,n}(0)x^{(n)} + \sum_{k=1}^{n-1} x^{(k)} c_{n,k}(0) \quad \text{[since } f(0) = 1\].
with \( c_{n,n}(0) = [f'(0)]^n \neq 0 \). Therefore, \( c_{n,n}(0) x^{(n)} \) is a polynomial in \( x \) of degree \( n \). Furthermore, since \( \sum_{k=1}^{n-1} x^{(k)} c_{n,k}(0) \) is a polynomial in \( x \) of degree less than \( n \), \( D^n g_x(0) \) is a polynomial in \( x \) of degree \( n \).

(ii) Now we show that polynomial \( D^n g_x(0) \) is binoid. Let \( P_n(x) \) denote the polynomial \( D^n g_x(0) \). Then, using the Leibniz differentiation rule for the \( n \)-th derivative of a product, we obtain

\[
D^n P_n(x + y) = D^n g_{x+y}(0)
= D[f(t)^x f(t)^y]|_{t=0}
= \sum_{k=0}^{n} \binom{n}{k} D^k f(t)^x D^{n-k} f(t)^y|_{t=0}
= \sum_{k=0}^{n} \binom{n}{k} D^k g_x(0) D^{n-k} g_y(0)
= \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y).
\]

Below we give examples of some binoid polynomial families and their respective generating functions.

**Some Examples of Binoid Polynomial Families**

**Example 1.** Let \( P_n(x) = x^n \) so that the binoid family is \( \{1, x, x^2, x^3, \ldots \} \). The identity in (1) becomes the common binomial theorem. Here \( P_n(x) \) has a generating function \( g_{x} \) given by \( g_{x}(t) = e^{tx} \).

**Example 2.** Let \( P_n(x) = x(x-1) \cdots (x-n+1) = x^{(n)} \). The binoid polynomial family here is \( \{1, x, x^2 - x, x^3 - 3x^2 + 2x, \ldots \} \). Identity (1) becomes

\[
(x + y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.
\]

We note that Berge ([1], p. 76) utilizes the forward difference operator \( \Delta \) to get the same result, also known as Vandermonde's formula. The generating function for \( P_n(x) \) is given by \( g_{x}(t) = (1 + t)^x \).
Example 3. Let $P_n(x) = x(x + 1) \cdots (x + n - 1) = x^{[n]}$. The binomial identity here is given by

$$(x + y)^{[n]} = \sum_{k=0}^{n} \binom{n}{k} x^{[k]} y^{[n-k]}.$$ 

This identity is also given by Berge. The generating function for $x^{[n]}$ is $g_x(t) = (1 - t)^{-x}$.

Example 4. Let $P_n(x) = x(x + n)^{n-1}$. The binoid polynomial family here is \{1, x, $x^2 + 2x$, $x^3 + 6x^2 + 9x$, \ldots\}. In this case, the binomial identity is

$$(x + y)(x + y + n)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} x^{(x + k)^{k-1}} y(y + n - k)^{n-k-1}.$$ 

The generating function is $g_x(t) = e^{\lambda x}$, where $\lambda$ is a function of $t$ defined implicitly by $\lambda e^{-\lambda} = t$ for $t \leq e^{-1}$. Equivalently, we can write $g_x(t) = e^{-xW(-t)}$ where $W$ is Lambert’s function, defined implicitly by $We^W = t$. A derivation for this generating function is provided in the appendix. For an alternate route see [2], p. 15, where a proof by Consul (that certain types of generalized Poisson random variables are closed under convolution) contains the binomial identity in question.

Example 5. Let $P_0(x) = 1$ and, for $n > 0$, let $P_n(x) = \sum_{i=1}^{n} S_{n,i} \cdot x^i$, where the $S_{n,i}$ are Stirling numbers of the second kind. (For a closed form expression of $S_{n,k}$ see [1], p.79.) Here identity (1) becomes

$$\sum_{i=1}^{n} S_{n,i} \cdot (x + y)^i = \sum_{k=0}^{n} \binom{n}{k} \sum_{i=1}^{k} S_{k,i} \cdot x^i \sum_{j=1}^{n-k} S_{n-k,j} \cdot y^j.$$ 

Here the generating function for $P_n(x)$ is given by $g_x(t) = \exp(x(e^t - 1))$. We note that $P_n(1) = \sum_{k=1}^{n} S_{n,k} = n$-th Bell number.

REFERENCES