Notes on Doubly-Surjective Finite Functions

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I. Definition of a Doubly-Surjective Function

A function \( f : D \rightarrow C \) is finite if the domain \( D \) and the codomain \( C \) are finite sets. A finite function \( f : D \rightarrow C \) is surjective if, for every \( y \) in the codomain \( C \), the cardinality of the pre-image set of \( y \) under \( f \) is at least one. In other words, \( f \) is surjective if \( |f^{-1}(y)| \geq 1 \) for all \( y \in C \). We extend the surjective concept and say a function \( f : D \rightarrow C \) is doubly-surjective if \( |f^{-1}(y)| \geq 2 \) for all \( y \in C \).

II. Double-Surjectiveness is a Consistent Property

The doubly-surjective property is consistent, which means (i) the union of completely-disjoint doubly-consistent finite functions is a doubly-surjective function, and (ii) a codomain induced partition of a doubly-surjective finite function yields a set of completely-disjoint doubly-consistent finite functions. [See Walsh's Toy Stories and Combinatorial Identities at http://capone.mtsu.edu/dwalsh/ATHEORM4.pdf for a formal definition of the consistent property.]

III. A Binomial Identity for the Cardinalities of Sets of Doubly-Surjective Functions

The consistent property implies the size, or cardinality, \( s(n, k) \) of the set \( F_{n,k} = \{ f : D \rightarrow C \text{ such that } f \text{ is doubly-surjective, } |D| = n, \text{ and } |C| = k \} \) satisfies the binomial identity

\[
s(n, k) = s(n, k_1 + k_2) = \sum_{j=0}^{n} \binom{n}{j} s(j, k_1) \cdot s(n-j, k_2)
\]

for positive integers \( k_1 \) and \( k_2 \) with \( k = k_1 + k_2 \).
IV. The Exponential Generating Function for Cardinalities

**Theorem 1.** If \( G_k \) denotes the exponential generating function for \( s(n, k) \), with positive integer \( k \) fixed, then

\[
G_k(t) = [G_1(t)]^k. \tag{2}
\]

**Proof (by induction).** For \( k = 1 \), the identity in (2) is trivial. Now, assume (2) holds for some arbitrary positive integer \( k \). Then

\[
G_{k+1}(t) = \sum_{n=0}^{\infty} s(n, 1 + k) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{j} s(j, 1) \cdot s(n - j, k) \right) \frac{t^n}{n!} \tag{by identity (1)}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \binom{n}{n-j} s(j, 1) \cdot s(n - j, k) \right) \frac{t^{n+j}}{n!}
\]

\[
= \sum_{j=0}^{\infty} s(j, 1) \frac{t^j}{j!} \sum_{n=j}^{\infty} s(n - j, k) \frac{t^{n-j}}{(n-j)!} \tag{by switching the order of summation}
\]

\[
= G_1(t) G_k(t)
\]

\[
= G_1(t) [G_1(t)]^k \tag{by the induction hypothesis}
\]

\[
= [G_1(t)]^{k+1}
\]

**Theorem 2.** The exponential generating function \( G_1 \) for \( s(n, 1) \) is given by

\[
G_1(t) = e^t - t - 1. \tag{3}
\]

**Proof.** For \(|D| = n\) and \(|C| = 1\), there is only one doubly-surjective function \( f : D \rightarrow C \) when \( n \geq 2 \) and none when \( n = 0 \) or 1. Hence

\[
s(n, 1) = \begin{cases} 
1 & \text{for } n \geq 2 \\
0 & \text{otherwise}
\end{cases}
\]
Therefore, the exponential generating function \( G_1 \) for \( s(n, 1) \) is given by

\[
G_1(t) = \sum_{n=0}^{\infty} s(n, 1) \frac{t^n}{n!} = \sum_{n=2}^{\infty} \frac{t^n}{n!} = e^t - t - 1.
\]

Thus we have the following theorem.

**Theorem 3.** Let \( s(n, k) \) denote the cardinality of the set

\[
F_{n,k} = \{ f : D \to C \text{ such that } f \text{ is doubly-surjective, } |D| = n, \text{ and } |C| = k \}.
\]

Then, for fixed positive integer \( k \), the generating function \( G_k \) for \( s(n, k) \) is given by

\[
G_k(t) = (e^t - t - 1)^k.
\] (4)

Hence, from Theorem 3, we obtain

\[
s(n, k) = D^n_t (e^t - t - 1)^k|_{t=0}.
\] (5)

**Example 1.** We use (5) to generate \( s(n, 2) \) with the Maple code below.

```maple
g := (t, r) -> (exp(t) - t - 1)^r;
g := \mathrm{(t}, r) \mapsto (\exp(t) - t - 1)^r
> seq(eval(diff(G(t,2),t$n),t=0),n=4..40);

6, 20, 50, 112, 238, 492, 1002, 2024, 4070, 8164, 16354, 32736, 65502,
131036, 262106, 524248, 1048534, 2097108, 4194258, 8388560,
16777166, 33554380, 67108810, 134217672, 268435398, 536870852,
1073741762, 2147483584, 4294967230, 8589934524
```
The sequence $s(n, 2)$ appears in the *On-Line Encyclopedia of Integer Sequences* as sequence A052515, the number of pairs of complementary sets of cardinality at least 2. It is also described as "the number of binary sequences of length $n$ having at least two 0's and at least two 1's. [From Geoffrey Critzer, Feb 11 2009]"

**Example 2.** We use (5) to generate $s(n, 3)$ with the *Maple* code below.

```
> G := (t, r) -> (exp(t) - t - 1)^r;

> seq(eval(diff(G(t, 3), t$n), t=0), n=6..30);
```

90, 630, 11508, 40950, 137610, 445896, 1410552, 4390386, 13514046, 41278068, 125405532, 379557198, 1145747538, 3452182656, 10380002848, 31230066186, 93828607686, 281775226860, 845929656900, 2539047258150, 7619759016090, 22864712861880, 68605412870088, 205839592489890

Besides counting doubly-surjective functions with codomain of size 3, $s(n, 3)$ also counts the number of ways to place $n$ numbered balls in 3 labeled boxes so that each box has at least 2 balls.

```
> seq(eval(diff(G(t, 4), t$n), t=0), n=8..30);
```

2520, 30240, 226800, 1367520, 7271880, 35692800, 165957792, 742822080, 3234711480, 13803744864, 58021888080, 241116750624, 993313349544, 4064913201216, 16549636147968, 67112688842496, 271323921459096, 1094303232174240, 4405390451382960, 17709538489849440, 71112371063277960, 285302897262913920, 1143863895743633760

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V. Deriving $s(n,k)$ Using Inclusion/Exclusion

We now present a formula for $s(n, k)$ derived using the principle of inclusion/exclusion.

**Theorem 4.**

$$s(n, k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{i=0}^{j} \left( \frac{n!}{(n-j)!} \right) (k-j)^{n-i}$$ (6)

for $1 \leq k \leq n/2$ and $n \geq 2$.

**Proof.**

For $j = 1, \ldots, k$, let $A_i$ denote the set of all non-doubly-surjective functions $f : D \rightarrow C$ with $|f^{-1}(i)| \leq 1$. For example, $A_1$ contains all functions $f$ which map at most one element to 1, $A_2$ contains functions which map at most one element to 2, etc. Similarly, for $i \neq j$, let $A_{ij}$ denote the set of all non-doubly-surjective functions with $|f^{-1}(i)| \leq 1$ and $|f^{-1}(j)| \leq 1$. In general, let $A_{i_1 i_2 \ldots i_j}$ denote the set of all non-doubly-surjective functions with $|f^{-1}(i_1)| \leq 1$, $|f^{-1}(i_2)| \leq 1, \ldots$, and $|f^{-1}(i_j)| \leq 1$.

We note that $\bigcup A_i \supseteq \bigcup A_{ij} \supseteq \cdots \supseteq \bigcup A_{i_1 i_2 \ldots i_j} \supseteq \cdots \supseteq A_{12 \ldots k}$. Furthermore, if $A$ denotes the set of all non-doubly-surjective functions, then

$$|A| = \sum_{i=1}^{k} |A_i| - \sum_{i<j} |A_{ij}| + \sum_{i<j<k} |A_{ijk}| - \ldots + (-1)^{j+1}\sum_{i_1<i_2<\ldots<i_j} |A_{i_1 i_2 \ldots i_j}| \pm \ldots (-1)^{k+1}|A_{12 \ldots k}|$$

$$= \binom{k}{1} |A_1| - \binom{k}{2} |A_{12}| + \binom{k}{3} |A_{123}| - \ldots (-1)^{j+1}\binom{k}{j} |A_{i_1 i_2 \ldots i_j}| \pm \ldots (-1)^{k+1}\binom{k}{k} |A_{12 \ldots k}|$$

$$= \sum_{j=1}^{k} (-1)^{j+1}\binom{k}{j} |A_{i_1 i_2 \ldots i_j}|$$ (7)

by the inclusion/exclusion principle.

To find $|A_{i_1 i_2 \ldots i_m}|$, we will list the steps in constructing a function in the set $A_{i_1 i_2 \ldots i_j}$ after first fixing the number $i$ of elements from $\{i_1, i_2, \ldots, i_j\}$ which have a pre-image of size 1. After multiplying the number of ways to take each step for fixed $i$, we sum the product as $i$ runs from 0 to $j$ to obtain the cardinality of $A_{i_1 i_2 \ldots i_j}$.

**Step 1.** Choose $i$ elements from $\{i_1, i_2, \ldots, i_j\}$ which have a pre-image of size 1. [Number of ways = $\binom{j}{i}$]

**Step 2.** Choose $i$ elements from $D$ that are mapped to the $i$ elements of $\{i_1, i_2, \ldots, i_j\}$ which have a pre-image of size 1. [Number of ways = $\binom{n}{i}$]
Step 3. Map the \(i\) chosen elements from \(D\) onto the \(i\) chosen elements of \(\{i_1, i_2, \ldots, i_j\}\). [Number of ways = \(i!\).]

Step 4. Map the remaining elements of \(D\) to \(C\setminus\{i_1, i_2, \ldots, i_j\}\). [Number of ways = \((k - j)^{n-i}\).]

Therefore

\[
|A_{i_1,i_2,\ldots,i_m}| = \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i! (k - j)^{n-i}
\]

(8)

Hence, using (8) in (7), we obtain

\[
|A| = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i! (k - j)^{n-i}
\]

(9)

Finally to derive \(s(n, k)\), we note that \(s(n, k) = k^n - |A|\) and so using (3) we obtain

\[
s(n, k) = k^n - \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i! (k - j)^{n-i}
\]

\[
= k^n + \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i! (k - j)^{n-i}
\]

\[
= \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i! (k - j)^{n-i}
\]

(\(\square\))

VI. The Sequence \(s(n, 3)\)

The sequence \(s(n, 3)\) can be interpreted in various ways:
(i) the number of doubly-surjective functions from a set of size \(n\) onto a set of size 3;
(ii) the number of ways to distribute \(n\) different toys to 3 different children so that each child gets at least 2 toys;
(iii) the number of ways to put \(n\) numbered balls into 3 labeled boxes so that each box gets at least 2 balls;
(iv) the number of length-\(n\) words that can be made using the letters A, B, and C with each letter occurring at least twice;
(v) the number of \(n\)-digit positive integers consisting of the digits 1, 2, and 3 with each of these digits appearing at least twice.

For example, \(s(6, 3) = 90\) since there are 90 six-digit integers satisfying the criteria. The first 30 of the ninety, namely those with initial digit 1, are given below:
Formulas for $s(n, 3)$

For $n \geq 6$, using the binomial identity in (1) and the fact that $s(n, 2) = 2^n - 2n - 2$, we obtain

\[
s(n, 3) = s(n, 2 + 1) = \sum_{k=4}^{n-2} \binom{n}{k} s(k, 2) \cdot s(n - k, 1)
\]

\[
= \sum_{k=4}^{n-2} \binom{n}{k} (2^k - 2k - 2)(1)
\]

\[
= 3^n - 3(2^n) - \frac{3}{2} n 2^n + 3n^2 + 3n + 3.
\]

(10)

Also, for $n \geq 6$, we obtain the following formula by using a multinomial identity implied by the binomial identity:

\[
s(n, 3) = s(n, 1 + 1 + 1) = \sum_{<i,j,k>} \frac{n!}{i!j!k!}
\]

(11)

where $2 \leq i, j, k$ and $i + j + k = n$; or, equivalently,

\[
s(n, 3) = \sum_{j=2}^{n-4} \sum_{i=2}^{n-2-j} \frac{n!}{i!j!(n-i-j)!}.
\]

(12)

We will find the initial terms in the sequences $s(n, 3)$ using the formula in (10).

The *Maple* code below is used.

```maple
> a:=n->3^n-3*2^n-3*n*2^(n-1)+3+3*n+3*n^2;
> seq(a(n),n=6..40);
```
The sequence $s(n, 4)$ can be interpreted in various ways:

(i) the number of doubly-surjective functions from a set of size $n$ onto a set of size $4$;
(ii) the number of ways to distribute $n$ different toys to 4 different children so that each child gets at least 2 toys;
(iii) the number of ways to put $n$ numbered balls into 4 labeled boxes so that each box gets at least 2 balls;
(iv) the number of length-$n$ words that can be made using the letters A, B, C, and D with each letter occurring at least twice;
(v) the number of $n$-digit positive integers consisting of the digits 1, 2, 3, and 4 with each of these digits appearing at least twice.

Formulas for $s(n, 4)$

For $n \geq 8$, using the binomial identity in (1) and the fact that

$$s(n, 3) = 3^n - 3(2^n) - \frac{3}{2} n 2^n + 3n^2 + 3n + 3,$$

we obtain

$$s(n, 4) = s(n, 3 + 1)$$

$$= \sum_{k=6}^{n-2} \binom{n}{k} s(k, 3) \cdot s(n-k, 1)$$

$$= \sum_{k=6}^{n-2} \binom{n}{k} (3^k - 3(2^k) - \frac{3}{2} k 2^k + 3k^2 + 3k + 3)(1)$$

$$= 4^n - 4(3^n) - \frac{4}{3} n(3^n) + 6(2^n) + \frac{9}{2} n(2^n) + \frac{3}{2} n^2(2^n) - 4 - 8n - 4n^3 \quad (13)$$
Also, for \( n \geq 8 \), we obtain the following formula by using a multinomial identity implied by the binomial identity:

\[
s(n, 4) = s(n, 1 + 1 + 1 + 1) = \sum_{<i,j,k,r>} \frac{n!}{i!j!k!r!}
\]  

(14)

where \( 3 \leq i, j, k \) and \( i + j + k + r = n \).

The following terms for \( s(n, 4) \) for \( n = 8 \) through \( n = 35 \) were obtained by Maple.

\[
> \text{seq}(s(n,4), n=8..35);
\]

2520, 30240, 226800, 1367520, 7271880, 35692800, 165957792, 742822080, 3234711480, 13803744864, 58021888080, 241116750624, 993313349544, 4064913201216, 16549636147968, 67112688842496, 271323921459096, 1094303232174240, 4405390451382960, 17709538489849440, 71112371063277960, 285302897262913920, 1143863895743633760, 4583688232518945600, 18360277039544528760, 7352015674644447840, 294325196736994816272, 1178056757852898287520,