Notes on Doubly-Surjective Finite Functions

Dennis Walsh
Middle Tennessee State University

I. Definition of a Doubly-Surjective Function

A function $f : D \to C$ is *finite* if the domain D and the codomain C are finite sets. A finite function $f : D \to C$ is *surjective* if, for every y in the codomain C, the cardinality of the pre-image set of y under f is at least one. In other words, f is surjective if $|f^{-1}(y)| \geq 1$ for all $y \in C$. We extend the surjective concept and say a function $f : D \to C$ is *doubly-surjective* if $|f^{-1}(y)| \geq 2$ for all $y \in C$.

II. Double-Surjectiveness is a Consistent Property

The doubly-surjective property is *consistent*, which means (i) the union of completely-disjoint doubly-consistent finite functions is a doubly-surjective function, and (ii) a codomain induced partition of a doubly-surjective finite function yields a set of completely-disjoint doubly-consistent finite functions. [See Walsh's *Toy Stories and Combinatorial Identities* at http://capone.mtsu.edu/dwalsh/ATHEORM4.pdf for a formal definition of the consistent property.]

III. A Binomial Identity for the Cardinalities of Sets of Doubly-Surjective Functions

The consistent property implies the size, or cardinality, $s(n,k)$ of the set $F_{n,k} = \{f : D \to C$ such that f is doubly-surjective, $|D| = n,$ and $|C| = k\}$ satisfies the binomial identity

$$s(n,k) = s(n, k_1 + k_2) = \sum_{j=0}^{n} \binom{n}{j} s(j, k_1) \cdot s(n-j, k_2) \qquad (1)$$

for positive integers k_1 and k_2 with $k = k_1 + k_2$.

IV. The Exponential Generating Function for Cardinalities

Theorem 1. If G_k denotes the exponential generating function for $s(n,k)$, with positive integer k fixed, then

$$G_k(t) = [G_1(t)]^k. \tag{2}$$

***Proof* (by induction)**. For $k=1$, the identity in (2) is trivial. Now, assume (2) holds for some arbitrary positive integer k. Then

$$G_{k+1}(t) = \sum_{n=0}^{\infty} s(n,1+k)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} s(j,1)\cdot s(n-j,k)\right)\frac{t^n}{n!} \quad \text{[by identity (1)]}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{n-j} s(j,1)\cdot s(n-j,k)\right)\frac{t^{j+n-j}}{n!}$$

$$= \sum_{j=0}^{\infty} s(j,1)\frac{t^j}{j!} \sum_{n=j}^{\infty} s(n-j,k)\frac{t^{n-j}}{(n-j)!} \quad \text{[by switching the order of summation]}$$

$$= G_1(t)\, G_k(t)$$

$$= G_1(t)[G_1(t)]^k \quad \text{[by the induction hypothesis]}$$

$$= [G_1(t)]^{k+1}$$

□

Theorem 2. The exponential generating function G_1 for $s(n,1)$ is given by

$$G_1(t) \;=\; e^t - t - 1. \tag{3}$$

Proof. For $|D| = n$ and $|C| = 1,$ there is only one doubly-surjective function $f : D \to C$ when $n \geq 2$ and none when $n = 0$ or 1. Hence

$$s(n,1) = \begin{cases} 1 \text{ for } n \geq 2 \\ 0 \;\; \text{otherwise} \end{cases} .$$

Therefore, the exponential generating function G_1 for $s(n,1)$ is given by

$$G_1(t) = \sum_{n=0}^{\infty} s(n,1)\, t^n/n!$$

$$= \sum_{n=2}^{\infty} t^n/n!$$

$$= e^t - t - 1. \qquad \square$$

Thus we have the following theorem.

Theorem 3 . Let $s(n,k)$ denote the cardinality of the set

$$F_{n,k} = \{f : D \to C \text{ such that } f \text{ is doubly-surjective, } |D| = n, \text{ and } |C| = k\}.$$

Then, for fixed positive integer k, the generating function G_k for $s(n,k)$ is given by

$$G_k(t) = (e^t - t - 1)^k. \tag{4}$$

Hence, from Theorem 3, we obtain

$$s(n,k) = D_t^n \, (e^t - t - 1)^k_{|_{t=0}}. \tag{5}$$

Example 1. We use (5) to generate $s(n,2)$ with the *Maple* code below.

```
>G:=(t,r)->(exp(t)-t-1)^r;
                                                 r
                   G := (t, r) -> (exp(t) - t - 1)

> seq(eval(diff(G(t,2),t$n),t=0),n=4..40);

  6,20,50,112,238,492,1002,2024,4070,8164,16354,32736,65502,
  131036,262106,524248,1048534,2097108,4194258,8388560,
  16777166,33554380,67108810,134217672,268435398,536870852,
  1073741762,2147483584,4294967230,8589934524
```

The sequence $s(n,2)$ appears in the *On-Line Encyclopedia of Integer Sequences* as sequence A052515, the number of pairs of complementary sets of cardinality at least 2. It is also described as "the number of binary sequences of length n having at least two 0's and at least two 1's. [From Geoffrey Critzer, Feb 11 2009] "

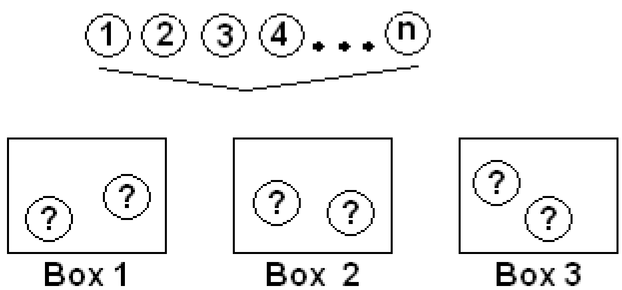
Example 2. We use (5) to generate $s(n,3)$ with the *Maple* code below.

```
>G:=(t,r)->(exp(t)-t-1)^r;
                                                    r
                    G := (t, r) -> (exp(t) - t - 1)

> seq(eval(diff(G(t,3),t$n),t=0),n=6..30);

  90, 630, 2940, 11508, 40950, 137610, 445896, 1410552, 4390386,
  13514046, 41278068, 125405532, 379557198, 1145747538,
  3452182656, 10388002848, 31230066186, 93828607686,
  281775226860, 845929656900, 2539047258150, 7619759016090,
  22864712861880, 68605412870088, 205839592489890
```

Besides counting doubly-surjective functions with codomain of size 3, $s(n,3)$ also counts the number of ways to place n numbered balls in 3 labeled boxes so that each box has at least 2 balls



Example 3. We use (5) to generate $s(n,4)$ with the *Maple* code below.

```
>G:=(t,r)->(exp(t)-t-1)^r;
                                                    r
                    G := (t, r) -> (exp(t) - t - 1)

> seq(eval(diff(G(t,4),t$n),t=0),n=8..30);

  2520, 30240, 226800, 1367520, 7271880, 35692800, 165957792,
  742822080, 3234711480, 13803744864, 58021888080, 241116750624,
  993313349544, 4064913201216, 16549636147968, 67112688842496,
  271323921459096, 1094303232174240, 4405390451382960,
  17709538489849440, 71112371063277960, 285302897262913920,
  1143863895743633760
```

V. Deriving s(n,k) Using Inclusion/Exclusion

We now present a formula for $s(n,k)$ derived using the principle of inclusion/exclusion.

Theorem 4.

$$s(n,k) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \frac{n!}{(n-i)!}(k-j)^{n-i} \tag{6}$$

for $1 \leq k \leq n/2$ and $n \geq 2$.

Proof.

For $j = 1, ..., k$, let A_i denote the set of all non-doubly-surjective functions $f : D \to C$ with $|f^{-1}(i)| \leq 1$. For example, A_1 contains all functions f which map at most one element to 1, A_2 contains functions which map at most one element to 2, etc. Similarly, for $i \neq j$, let A_{ij} denote the set of all non-doubly-surjective functions with $|f^{-1}(i)| \leq 1$ and $|f^{-1}(j)| \leq 1$. In general, let $A_{i_1 i_2 ... i_j}$ denote the set of all non-doubly-surjective functions with $|f^{-1}(i_1)| \leq 1$, $|f^{-1}(i_2)| \leq 1$,..., and $|f^{-1}(i_j)| \leq 1$.

We note that $\cup A_i \supseteq \cup A_{ij} \supseteq \cdots \supseteq \cup A_{i_1 i_2 ... i_j} \supseteq \cdots \supseteq A_{12..k}$. Furthermore, if A denotes the set of all non-doubly-surjective functions, then

$$|A| = \sum_{i=1}^{k}|A_i| - \sum_{i<j}|A_{ij}| + \sum_{i<j<k}|A_{ijk}| - ...(-1)^{j+1}\sum_{i_1<i_2<..<i_j}|A_{i_1 i_2 ... i_j}| \pm ....(-1)^{k+1}|A_{12..k}|$$

$$= \binom{k}{1}|A_i| - \binom{k}{2}|A_{ij}| + \binom{k}{3}|A_{ijk}| - ...(-1)^{j+1}\binom{k}{j}|A_{i_1 i_2 ... i_j}| \pm ....(-1)^{k+1}\binom{k}{k}|A_{12..k}|$$

$$= \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}|A_{i_1 i_2 ... i_j}| \tag{7}$$

by the inclusion/exclusion principle.

To find $|A_{i_1 i_2 ... i_m}|$, we will list the steps in constructing a function in the set $A_{i_1 i_2 ... i_j}$ after first fixing the number i of elements from $\{i_1, i_2, ..., i_j\}$ which have a pre-image of size 1. After multiplying the number of ways to take each step for fixed i, we sum the product as i runs from 0 to j to obtain the cardinality of $A_{i_1 i_2 ... i_j}$.

Step 1. Choose i elements from $\{i_1, i_2, ..., i_j\}$ which have a pre-image of size 1. [Number of ways $= \binom{j}{i}$.]
Step 2. Choose i elements from D that are mapped to the i elements of $\{i_1, i_2, ..., i_j\}$ which have a pre-image of size 1. [Number of ways $= \binom{n}{i}$.]

Step 3. Map the i chosen elements from D onto the i chosen elements of $\{i_1,\ i_2,\ ...,\ i_j\}$.[Number of ways $= i!$.]

Step 4. Map the remaining elements of D to $C\backslash\{i_1,\ i_2,\ ...,\ i_j\}$. [Number of ways $= (k-j)^{n-i}$.]

Therefore

$$|A_{i_1 i_2 \dots i_m}| = \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i!\, (k-j)^{n-i}$$

(8)

Hence, using (8) in (7), we obtain

$$|A| = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i!\, (k-j)^{n-i} \tag{9}$$

Finally to derive $s(n,k)$, we note that $s(n,k) = k^n - |A|$ and so using (3) we obtain

$$s(n,k) = k^n - \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i!\, (k-j)^{n-i}$$

$$= k^n + \sum_{j=1}^{k} (-1)^{j} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i!\, (k-j)^{n-i}$$

$$= \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{i} i!\, (k-j)^{n-i}$$ □

VI. The Sequence $s(n,3)$

The sequence $s(n,3)$ can be interpreted in various ways:
(i) the number of doubly-surjective functions from a set of size n onto a set of size 3;
(ii) the number of ways to distribute n different toys to 3 different children so that each child gets at least 2 toys;
(iii) the number of ways to put n numbered balls into 3 labeled boxes so that each box gets at least 2 balls;
(iv) the number of length-n words that can be made using the letters A, B, and C with each letter occurring at least twice;
(v) the number of n-digit positive integers consisting of the digits 1, 2, and 3 with each of these digits appearing at least twice.

For example, $s(6,3) = 90$ since there are 90 six-digit integers satisfying the criteria. The first 30 of the ninety, namely those with initial digit 1, are given below:

112233, 112323, 112332, 113223, 113232, 113322,
121233, 121323, 121332, 122133, 122313, 122331,
123123, 123132, 123213, 123231, 123312, 123321,
131223, 131232, 131322, 132123, 132132, 132213,
132231, 132312, 132321, 133122, 133212, 133221

Formulas for $s(n,3)$

For $n \geq 6$, using the binomial identity in (1) and the fact that $s(n,2) = 2^n - 2n - 2$, we obtain

$$\begin{aligned} s(n,3) &= s(n,2+1) \\ &= \sum_{k=4}^{n-2} \binom{n}{k} s(k,2) \cdot s(n-k,1) \\ &= \sum_{k=4}^{n-2} \binom{n}{k} (2^k - 2k - 2)(1) \\ &= 3^n - 3(2^n) - \tfrac{3}{2} n\, 2^n + 3n^2 + 3n + 3. \qquad (10) \end{aligned}$$

Also, for $n \geq 6$, we obtain the following formula by using a multinomial identity implied by the binomial identity:

$$s(n,3) = s(n,1+1+1) = \sum_{<i,j,k>} \frac{n!}{i!j!k!} \qquad (11)$$

where $2 \leq i,j,k$ and $i+j+k=n$; or, equivalently,

$$s(n,3) = \sum_{j=2}^{n-4} \sum_{i=2}^{n-2-j} \frac{n!}{i!j!(n-i-j)!}. \qquad (12)$$

We will find the initial terms in the sequences $s(n,3)$ using the formula in (10).

The *Maple* code below is used.

```
> a:=n->3^n-3*2^n-3*n*2^(n-1)+3+3*n+3*n^2;

> seq(a(n),n=6..40);
```

```
90, 630, 2940, 11508, 40950, 137610, 445896, 1410552, 4390386,
13514046, 41278068, 125405532, 379557198, 1145747538,
3452182656, 10388002848, 31230066186, 93828607686,
281775226860, 845929656900, 2539047258150, 7619759016090,
22864712861880, 68605412870088, 205839592489890,
617567095846350, 1852801145522916, 5558609594992812,
16676253986734206, 50029638133524066, 150090718286829168,
450275865712223472, 1350835224998579898, 4052521343036426838,
12157596189824383836
```

VII. The Sequence $s(n, 4)$

The sequence $s(n, 4)$ can be interpreted in various ways:
(i) the number of doubly-surjective functions from a set of size n onto a set of size 4;
(ii) the number of ways to distribute n different toys to 4 different children so that each child gets at least 2 toys;
(iii) the number of ways to put n numbered balls into 4 labeled boxes so that each box gets at least 2 balls;
(iv) the number of length-n words that can be made using the letters A, B, C, and D with each letter occurring at least twice;
(v) the number of n-digit positive integers consisting of the digits 1, 2, 3, and 4 with each of these digits appearing at least twice.

Formulas for $s(n, 4)$

For $n \geq 8$, using the binomial identity in (1) and the fact that

$$s(n,3) = 3^n - 3(2^n) - \tfrac{3}{2}n\,2^n + 3n^2 + 3n + 3,$$

we obtain

$$\begin{aligned} s(n,4) = {} & s(n, 3+1) \\ & = \sum_{k=6}^{n-2} \binom{n}{k} s(k,3) \cdot s(n-k,1) \\ & = \sum_{k=6}^{n-2} \binom{n}{k} (3^k - 3(2^k) - \tfrac{3}{2}k\,2^k + 3k^2 + 3k + 3)(1) \\ & = 4^n - 4(3^n) - \tfrac{4}{3}n(3^n) + 6(2^n) + \tfrac{9}{2}n(2^n) + \tfrac{3}{2}n^2(2^n) - 4 - 8n - 4n^3 \end{aligned} \tag{13}$$

Also, for $n \geq 8,$ we obtain the following formula by using a multinomial identity implied by the binomial identity:

$$s(n,4) = s(n,1+1+1+1) = \sum_{<i,j,k,r>} \frac{n!}{i!j!k!r!} \tag{14}$$

where $3 \leq i, j, k$ and $i + j + k + r = n.$

The following terms for $s(n, 4)$ for $n = 8$ through $n = 35$ were obtained by *Maple*.

```
>seq(s(n,4),n=8..35);

  2520, 30240, 226800, 1367520, 7271880, 35692800, 165957792,
  742822080, 3234711480, 13803744864, 58021888080, 241116750624,
  993313349544, 4064913201216, 16549636147968, 67112688842496,
  271323921459096, 1094303232174240, 4405390451382960,
  17709538489849440, 71112371063277960, 285302897262913920,
  1143863895743633760, 4583688232518945600,
  18360277039544528760, 73520156746446447840,
  294325196736994816272, 1178056757852898287520,
```