Determination of a Statistically-Significant Lead in an Election Poll

Dennis Walsh
Middle Tennessee State University, Fall 2013

I. Introduction

In election season, most media outlets present poll results in an abbreviated form. For example, an NBC/Wall Street Journal presidential poll on November 1, 2008 provided the following results: sample size, \( n = 1011 \); proportion for Obama, \( \hat{p}_O = .51 \); and proportion for McCain, \( \hat{p}_M = .43 \). Assuming the poll represented a true random sample of likely voters, did Obama have a statistically significant lead at that time?

We will provide a formula to calculate a confidence interval for the difference in the true proportions, and then we will derive an easy calculation to determine if statistical significance is obtained for a significance level of approximately .05.

II. Large-Sample Confidence Interval for the Difference of Proportions (Based on a Single Sample)

For \( i = 1, 2 \), let \( p_i \) denote the probability that a random voter supports candidate \( i \). If \( x \) voters support candidate 1 and \( y \) voters support candidate 2 in a random sample of size \( n \), then a confidence interval for the difference \( p_1 - p_2 \) is given by

\[
\left( \frac{x}{n} - \frac{y}{n} - z_c \sqrt{\frac{n(x+y)-(x-y)^2}{n^3}}, \frac{x}{n} - \frac{y}{n} + z_c \sqrt{\frac{n(x+y)-(x-y)^2}{n^3}} \right),
\]

where \( z_c \) is the appropriate standard normal percentile for confidence level \( c \). For example, \( z_c = 1.960 \) for a 95\% confidence level. Equivalent to interval (1) is

\[
\left( \hat{p}_1 - \hat{p}_2 - z_c \sqrt{\frac{\hat{p}_1+\hat{p}_2-(\hat{p}_1-\hat{p}_2)^2}{n}}, \hat{p}_1 - \hat{p}_2 + z_c \sqrt{\frac{\hat{p}_1+\hat{p}_2-(\hat{p}_1-\hat{p}_2)^2}{n}} \right)
\]

where \( \hat{p}_1 = x/n, \hat{p}_2 = y/n \), and \( z_c \) denotes the standard normal quantile appropriate for confidence coefficient \( c \).

Derivation

For \( i = 1, 2 \), let \( p_i \) denote the probability that a random voter supports candidate \( i \), and, in a single random sample of size \( n \), let \( X_i \) denote the random number of voters who support candidate \( i \). Then \( (X_1, X_2) \) has a trinomial distribution with probability mass function \( f \) given by
\[ f(x, y) = P(X_1 = x, X_2 = y) = \frac{n!}{x!y!(n-x-y)!} p_1^x p_2^y (1 - p_1 - p_2)^{n-x-y} \]

for \( 0 \leq x \leq n - y, \ 0 \leq y \leq n - x \).

To construct the confidence interval for \( p_1 - p_2 \), we will need the variance of the difference \( X_1 - X_2 \), that is, \( \text{Var}(X_1 - X_2) \). We obtain

\[
\text{Var}(X_1 - X_2) = \text{Var}(X_1) + \text{Var}(X_2) - 2\text{Cov}(X_1, X_2) \\
= np_1(1 - p_1) + np_2(1 - p_2) - 2(-np_1p_2) \\
= np_1 + np_2 - n(p_1 - p_2)^2.
\]

In turn, the result above implies that

\[
\text{Var}(X_1/n - X_2/n) = \frac{p_1 + p_2 - (p_1 - p_2)^2}{n}.
\]

Therefore, if the sample size \( n \) is sufficiently large, Liapounov's version of the central limit theorem implies that \( \left( \frac{X_1}{n} - \frac{X_2}{n} \right) \) has an approximate normal distribution with mean \( p_1 - p_2 \) and variance \( \frac{p_1 + p_2 - (p_1 - p_2)^2}{n} \). Thus, if \( x \) voters support candidate 1 and \( y \) voters support candidate 2, then confidence interval for \( p_1 - p_2 \) is given by

\[
\left( \frac{x}{n} - \frac{y}{n} - z_c \sqrt{\frac{x+y}{n(n-x-y)}}, \ \frac{x}{n} - \frac{y}{n} + z_c \sqrt{\frac{x+y}{n(n-x-y)}} \right)
\]

or

\[
\left( \frac{x}{n} - \frac{y}{n} - z_c \sqrt{\frac{n(x+y)-(x-y)^2}{n^3}}, \ \frac{x}{n} - \frac{y}{n} + z_c \sqrt{\frac{n(x+y)-(x-y)^2}{n^3}} \right)
\]

or

\[
\left( \hat{p}_1 - \hat{p}_2 - z_c \sqrt{\frac{\hat{p}_1 + \hat{p}_2 - (\hat{p}_1 - \hat{p}_2)^2}{n}}, \ \hat{p}_1 - \hat{p}_2 - z_c \sqrt{\frac{\hat{p}_1 + \hat{p}_2 - (\hat{p}_1 - \hat{p}_2)^2}{n}} \right)
\]

where \( \hat{p}_1 = x/n \), \( \hat{p}_2 = y/n \), and \( z_c \) denotes the standard normal quantile appropriate for confidence coefficient \( c \).
Example. Using the poll results given in the introduction, an approximate 95% confidence interval for \( p_O - p_M \) is given by

\[
\left( .08 - 1.96\sqrt{(.51 + .43 - (.08)^2)/1011} , .08 + 1.96\sqrt{(.51 + .43 - (.08)^2)/1011} \right)
\]

\[
= (.08 - .06 , .08 + .06)
\]

\[
= (.02 , .14).
\]

In other words, we are about 95% certain that \(.02 \leq p_O - p_M \leq .14\).

III. Hypothesis Test at the .05 Level

In testing \( H_0 : p_1 = p_2 \) versus \( H_a : p_1 \neq p_2 \)
when using approximate level of significance .05 and when the sample size is large, we reject \( H_0 \) and conclude a statistically significant difference if

\[
\left| \frac{x}{n} - \frac{y}{n} \right| \geq 1.96\sqrt{\frac{\bar{p} + \bar{p} - (\bar{p} - \bar{p})^2}{n}}
\]

where \( \bar{p} \) is the best estimate of the null common value, that is, \( \bar{p} = \frac{1}{2} \left( \frac{x}{n} + \frac{y}{n} \right) \).

Hence,

\[
1.96\sqrt{\frac{\bar{p} + \bar{p} - (\bar{p} - \bar{p})^2}{n}} = 1.96\sqrt{\frac{\bar{p} + \bar{p}}{n}}
\]

\[
= 1.96\sqrt{\frac{2 \left( \frac{1}{2} \left( \frac{x}{n} + \frac{y}{n} \right) \right)}{n}}
\]

\[
= 1.96\frac{\sqrt{x+y}}{n}.
\]

Therefore, we reject \( H_0 \), if

\[
\left| \frac{x}{n} - \frac{y}{n} \right| \geq 1.96\frac{\sqrt{x+y}}{n},
\]

or, if

\[
|x - y| \geq 1.96\sqrt{x + y}.
\]

Squaring both sides, we obtain the following decision rule:

\[
\text{Reject } H_0 \text{ if } (x - y)^2 \geq 3.842(x + y).
\]
Rule of Thumb

For a rule of thumb, we may use

$$\text{Reject } H_0 \text{ if } (x - y)^2 \geq 4(x + y)$$

with an approximate level of significance $\alpha$ about .046.

An approximate $p$-value for the test is given by

$$2*P(Z \geq \frac{|x-y|}{\sqrt{x+y}}) \text{ where } Z \sim N(0, 1), \text{ standard normal.}$$

Example. Using the data in the introduction, since $.51(1011) \approx .516 = x$ and $.43(1011) \approx .435 = y$, we have

$$(x - y)^2 = (516 - 435)^2 = 81^2 = 6561$$

and

$$4(x + y) = 4(516 + 435) = 4(951) = 3804.$$ 

Therefore, using $\alpha = .046$, we reject $H_0$ and conclude a significant difference in proportions. Also,

$$p\text{-value} \approx 2*P(Z \geq \frac{81}{\sqrt{951}})$$

$$= 2(.0043121078)$$

$$\approx .0086.$$