I. Introduction

Finite geometric series arise in many areas of mathematics and science. When a product is preferred to a sum, the process of factoring the sum is important. The question we address is how can we factor \( 1 + r + r^2 + \ldots + r^{n-1} \)? Sometimes there are no factorizations. However, when \( n \) is composite, the series can always be factored. We will first look at a well-established factorization involving what are called cyclotomic polynomials. Then we explore an alternative route that uses a simple algorithm.

II. Factoring with Cyclotomic Polynomials

Consider the following finite geometric series \( \sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \ldots + r^{n-1} \).

When \( n \) is composite, there is a well known factorization of this series using cyclotomic polynomials:

\[
\sum_{k=0}^{n-1} r^k = \prod_{d|n} \Phi_d(r)
\]

(1)

where \( \Phi_k(r) \) is the \( k \)-th cyclotomic polynomial and the product is over all divisors \( d \) of \( n \) that are greater than 1. The \( k \)-th cyclotomic polynomial is an irreducible polynomial and has degree \( \phi(k) \), where \( \phi(k) \) denotes the number of positive integers less than \( k \) and relatively prime to \( k \).

To be precise, for \( k \geq 1 \), the \( k \)-th cyclotomic polynomial is given by

\[
\Phi_k(r) = \prod_{(j,n)=1} (r - \zeta_j)
\]

where \( \zeta_j = \exp(2\pi i j/n) \), a root of unity in \( \mathbb{C} \), and the product is over all positive integers \( j \) that are relatively prime to \( n \).

There is a way to obtain the cyclotomic polynomials without calculating the roots of unity. It arises from defining the cyclotomic polynomials recursively. Given \( \Phi_1(r) = r - 1 \), identity (1) is equivalent to

\[
r^n - 1 = \prod_{d|n} \Phi_d(r)
\]

(2)
where the product is now over all divisors of $n$. Hence

$$\Phi_n(r) = \frac{r^n-1}{\prod_{d|n} \Phi_d(r)}$$

where the product is over all divisors of $n$ that are less than $n$. Thus we obtain

$$\Phi_2(r) = \frac{r^2-1}{\Phi_1(r)} = \frac{r^2-1}{r-1} = r + 1,$$

$$\Phi_3(r) = \frac{r^3-1}{\Phi_1(r)} = \frac{r^3-1}{r-1} = r^2 + r + 1,$$

$$\Phi_4(r) = \frac{r^4-1}{\Phi_1(r)\Phi_2(r)} = \frac{r^4-1}{(r-1)(r+1)} = r^2 + 1,$$

$$\Phi_5(r) = \frac{r^5-1}{\Phi_1(r)} = \frac{r^5-1}{r-1} = r^4 + r^3 + r^2 + r + 1,$$

$$\Phi_6(r) = \frac{r^6-1}{\Phi_1(r)\Phi_2(r)\Phi_3(r)} = \frac{r^6-1}{(r-1)(r+1)(r^2+1)} = r^2 - r + 1,$$

We can continue in this fashion to get any cyclotomic polynomial we desire.

**Example.** To factor $1 + r + \ldots + r^{11}$, we need $\Phi_2(r)$, $\Phi_3(r)$, $\Phi_4(r)$, $\Phi_6(r)$ and $\Phi_{12}(r)$ because 2,3,4,6, and 12 are the non-unity divisors of 12. To obtain $\Phi_{12}(r)$ we can use the following short-cut formula:

$$\Phi_{pn}(r) = \Phi_n(r^p)$$

whenever prime $p$ divides $n$. Hence $\Phi_{12}(r) = \Phi_6(r^2) = r^4 - r^2 + 1$, and we get the factorization

$$1 + r + \ldots + r^{11} = \Phi_2(r) \Phi_3(r) \Phi_4(r) \Phi_6(r) \Phi_{12}(r)$$

$$= (r + 1)(r^2 + r + 1)(r^2 + 1)(r^2 - r + 1)(r^4 - r^2 + 1).$$

For a readily accessible introduction to cyclotomic polynomials, see the MathWorld internet site at http://mathworld.wolfram.com/CyclotomicPolynomial.html. Besides the definitions and results used above, the site contains a list of other references.

**III. An Alternate Route to Factoring a Finite Geometric Series**

To bypass the possible hassle in obtaining the cyclotomic polynomials, we present below a simple algorithm that produces multiple factorizations of $1 + r + \ldots + r^{n-1}$ when $n$ is composite. The factorizations are composed of factors that
are themselves finite geometric series. Furthermore, for each ordered factorization of \( n \), we will obtain a different factorization of the series.

We will introduce the algorithm with the following example. To obtain a factorization of \( 1 + r + \ldots + r^{29} \), we pick an ordered factorization of 30, say \( 30 = (5)(2)(3) \). Since we chose a factorization with 3 factors, we write \( 1 + r + \ldots + r^{29} = S_1 \cdot S_2 \cdot S_3 \). The number of terms in \( S_i \) (\( i = 1, 2, 3 \)) equals the \( i \)-th factor in our factorization of 30, and the common ratio of \( S_i \) is \( r^{\pi_i} \), where \( \pi_1 = 1 \) and \( \pi_i \) (\( i = 1, 2 \)) denotes a partial product of the factors of 30. Specifically, \( S_1 \) is the finite geometric series in \( r \) with 5 terms, \( S_2 \) is the finite geometric series in \( r^5 \) with 2 terms, and \( S_3 \) is the finite geometric series in \( r^{10} \) with 3 terms. Therefore,

\[
1 + r + \ldots + r^{29} = (1 + r + r^2 + r^3 + r^4)(1 + r^5)(1 + r^{10} + r^{20}).
\]

We note that the last two factors above are reducible, a concern that we will address later.

The well-known summation formula

\[
\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}
\]

(3)

will be used in the proof of our factorization result, which is formally given in the following theorem.

**Theorem.** Let \( n \) be a composite integer such that \( n = d_1 \cdot d_2 \cdot \ldots \cdot d_s \) where the integer divisors \( d_j \) satisfy \( 1 < d_j < n \). Define \( d_0 = 1 \), an let \( \pi_k \) denote the partial product of the \( d_j \)'s such that \( \pi_k = \prod_{j=0}^{k} d_j \) for \( k = 0, 1, \ldots, s \). Note that \( \pi_s = n \). The following identity holds all nonzero real \( r \),

\[
\sum_{k=0}^{n-1} r^k = \prod_{j=1}^{s} \sum_{i=0}^{d_{j-1} - 1} \left( r^{\pi_{j-1}} \right)^i.
\]

(4)

**Proof.** For the case \( r = 1 \), \( \sum_{k=0}^{n-1} 1 = n \) and \( \prod_{j=1}^{s} \sum_{i=0}^{d_{j-1} - 1} 1 = \prod_{j=1}^{s} d_j = \pi_s = n \).

For \( r \neq 1 \), after applying identity (3) to the left hand side of (4), we obtain

\[
\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}.
\]

On the other hand, upon applying identity (3) to the right hand side of (4), we get

\[
\prod_{j=1}^{s} \sum_{i=0}^{d_{j-1} - 1} \left( r^{\pi_{j-1}} \right)^i = \prod_{j=1}^{s} \frac{(r^{\pi_{j-1}})^{d_{j-1} - 1}}{(r^{\pi_{j-1}} - 1)}
\]
\[
= \prod_{j=1}^{n} \frac{(r^{x_j})-1}{(r^{x_j})^{-1}-1} = \frac{(r^{x_1})-1}{r-1} \cdot \frac{(r^{x_2})-1}{(r^{x_1})^{-1}-1} \cdots \frac{(r^{x_n})-1}{(r^{x_{n-1}})^{-1}-1} = \frac{r^n-1}{r-1}.
\]

**Example.** To factor \(1 + r + \ldots + r^{11}\), we pick a factorization of 12 to get a unique factorization of the sum.

<table>
<thead>
<tr>
<th>factorization of 12</th>
<th>factorization of (1 + r + \ldots + r^{11})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (2)(2)(3)</td>
<td>((1 + r)(1 + r^2)(1 + r^4 + r^8))</td>
</tr>
<tr>
<td>2. (2)(3)(2)</td>
<td>((1 + r)(1 + r^2 + r^4)(1 + r^6))</td>
</tr>
<tr>
<td>3. (3)(2)(2)</td>
<td>((1 + r + r^2)(1 + r^3)(1 + r^6))</td>
</tr>
<tr>
<td>4. (3)(4)</td>
<td>((1 + r + r^2)(1 + r^3 + r^6 + r^9))</td>
</tr>
<tr>
<td>5. (4)(3)</td>
<td>((1 + r + r^2 + r^3)(1 + r^4 + r^8))</td>
</tr>
<tr>
<td>6. (2)(6)</td>
<td>((1 + r)(1 + r^2 + r^4 + r^6 + r^8 + r^{10}))</td>
</tr>
<tr>
<td>7. (6)(2)</td>
<td>((1 + r + r^2 + r^3 + r^4 + r^5)(1 + r^6))</td>
</tr>
</tbody>
</table>

Note that (1) and (2) imply that \(\frac{(1+r^4+r^8)}{(1+r^2+r^4)} = \frac{(1+r^6)}{(1+r^2)} = 1 - r^2 + r^4\).

Hence \((1 + r^4 + r^8) = (1 + r^2 + r^4)(1 - r^2 + r^4)\).

Next (2) and (3) imply \(\frac{(1+r^2+r^4)}{(1+r+r^2)} = \frac{(1+r^3)}{(1+r)} = 1 - r + r^2\).

Hence \((1 + r^2 + r^4) = (1 + r + r^2)(1 - r + r^2)\).

Therefore, after substitution into factorization (1), we get

\[
1 + r + \ldots + r^{11} = (1 + r)(1 + r^2)(1 + r + r^2)(1 - r + r^2)(1 - r^2 + r^4),
\]
the factorization with irreducible cyclotomic polynomials.

IV. Some Factoring Facts

Using our algorithm, we often encounter a factor of the form $1 + r^m$. When can $1 + r^m$ be factored? The following well-known results answer this question.

Fact 1. For odd $m > 1$, 
$$1 + r^m = (1 + r)(1 - r + r^2 - \ldots \pm r^{m-1}).$$

Example: $1 + r^5 = (1 + r)(1 - r + r^2 - r^3 + r^4)$

However, $m$ does not have to be odd in order to factor $1 + r^m$. Consider $1 + r^6$ which can be written as $1 + (r^2)^3$. Hence $1 + r^6 = (1 + r^2)(1 - r^2 + r^4)$.

Fact 2. For $n = st$, where $s$ and $t$ are positive integers and $t$ is an odd integer,

$$1 + r^n = 1 + r^{st} = (1 + r^s)(1 - r^s + r^{2s} - \ldots \pm r^{s(t-1)}).$$

Example: $1 + r^{10} = (1 + r^2)(1 - r^2 + r^4 - r^6 + r^8)$

Note that Facts 1 and 2 imply that $1 + r^n$ can be factored whenever $n$ is not a power of 2.

Fact 3. For even $n$,

$$1 + r^2 + r^4 + \ldots + r^{2n} = (1 + r + r^2 + \ldots + r^n)(1 - r + r^2 - \ldots + r^n).$$

Example. $1 + r^2 + \ldots + r^8 = (1 + r + r^2 + r^3 + r^4)(1 - r + r^2 - r^3 + r^4)$

Note that $1 + r^2 + r^4 + \ldots + r^{2n}$ can be factored also when $n$ is odd. Suppose $n = 2m - 1$ with $m > 1$. Since the number of terms in the sum is $2k$, our theorem above applies and we get the following result.

Fact 4. For $m > 1$,

$$1 + r^2 + r^4 + \ldots + r^{2(2m-1)} = (1 + r^2)(1 + r^4 + r^8 + \ldots + r^{4(m-1)})$$

and

$$1 + r^2 + r^4 + \ldots + r^{2(2m-1)} = (1 + r^2 + r^4 + \ldots + r^{2(m-1)})(1 + r^{2m}).$$

Example. $1 + r^2 + r^4 + r^6 + r^8 + r^{10} = (1 + r^2)(1 + r^4 + r^8)$

and $1 + r^2 + r^4 + r^6 + r^8 + r^{10} = (1 + r^2 + r^4)(1 + r^6).$
But
\[(1 + r^4 + r^8) = (1 + r^2 + r^4)(1 - r^2 + r^4) = (1 + r + r^2)(1 - r + r^2)(1 - r^2 + r^4)\]
so that
\[1 + r^2 + r^4 + r^6 + r^8 + r^{10} = (1 + r^2)(1 + r + r^2)(1 - r + r^2)(1 - r^2 + r^4)\]

The theorem gives us the following result.

**Fact 5.** For odd \(n > 1\) with \(n = 2k - 1\), and any \(m\),

\[1 + r^m + r^{2m} + r^{3m} + \ldots + r^{nm} = (1 + r^m)(1 + r^{2m} + r^{4m} \ldots + r^{(n-1)m})\]
and

\[1 + r^m + r^{2m} + r^{3m} + \ldots + r^{nm} = (1 + r^m + r^{2m} + \ldots + r^{(k-1)m})(1 + r^{km}).\]

**Example.**

\[1 + r^5 + r^{10} + r^{15} + r^{20} + r^{25} = (1 + r^5)(1 + r^{10} + r^{20}) = (1 + r^5 + r^{10})(1 + r^{15})\]

Can \(1 + r^m + r^{2m} + r^{3m} + \ldots + r^{nm}\) be factored if \(n\) is even? If \((n + 1)\) is not prime, then the theorem gives us a factorization. What if \((n + 1)\) is prime? For example, can \(1 + r^3 + r^6 + r^9 + r^{12}\) be factored? Yes, it can. But \(1 + r^5 + r^{10} + r^{15} + r^{20}\) cannot.

**Conjecture.** Let \(p\) be a prime. The cyclotomic polynomial \(1 + r + r^2 + \ldots + r^{p-1}\) is a factor of \(1 + r^m + r^{2m} + \ldots + r^{(p-1)m}\) if and only if \(m\) is not a positive power of \(p\).

Conjecture.

**Investigate:** When can \(1 + r^n + r^{2n}\) be factored.

**Example.** Factor \(1 + r + \ldots + r^{29}\).