Notes on Functions from Subsets of \{1,2,...,n\} into \{1,2,...,n\}

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I. Functions from Subsets of \{1,2,...,n\} into \{1,2,...,n\}

Let \([n]\) denote the set of the first \(n\) positive integers, that is, \([n] = \{1, 2, \ldots, n\}\).

We will investigate functions \(f\) of the form \(f : D \to [n]\) where \(D\) is a subset of \([n]\). Let \(\mathcal{F}_n = \{f : D \to [n]\} \text{ where } D \text{ is a subset of } [n]\). We first find the cardinality \(c\) of \(\mathcal{F}_n\). Letting \(k\) denote the size of the domain \(D\), we have

\[
c(n) = |\mathcal{F}_n| = \sum_{k=0}^{n} \binom{n}{k} n^k = (n + 1)^n
\]

(1)

since there are \(\binom{n}{k}\) ways to choose the \(k\) elements of \(D\) and then there are \(n^k\) ways to map \(k\) elements into \([n]\). See the integer sequence A000169 in the On-Line Encyclopedia of Integer Sequences.

Each of the functions described above can be uniquely identified with a labeled directed graph on \(n\) vertices each of which has out-degree no greater than one. For example, \(f : \{1, 2, 4\} \to [4]\) defined by \(f(1) = 3\), \(f(2) = 3\), and \(f(4) = 4\) is identical to the graph

\[
\begin{array}{c}
1 \\
 \rightarrow \\
3 \\
 \leftarrow \\
2 \\
 \rightarrow \\
4
\end{array}
\]

Furthermore, each of the functions \(f\) can be uniquely associated with an \(n \times n\) binary matrix that has entry \((i, j)\) equal to 1 if \(f(i) = j\) and equal to 0 otherwise.

Example 1. Let \(f : \{1, 2, 4\} \to [4]\) be defined by \(f(1) = 3\), \(f(2) = 3\), and \(f(4) = 4\). This function is equivalent to the matrix

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Thus \((n + 1)^n\) is the number of \(n \times n\) binary matrices with each row sum equal to 0 or 1.
II. Injective Functions from Subsets of $[n]$ to $[n]$.

Let $\mathcal{F}_{I,n}$ denote the set $\{f : D \rightarrow [n] \mid D \text{ is a subset of } [n] \text{ and } f \text{ is injective}\}$. If subset $D$ has size $k$, then the number of injective functions from $D$ to $[n]$ is $n!/(n-k)!$. Summing over all subsets of $[n]$ we obtain the cardinality of $\mathcal{F}_{I,n}$:

$$|\mathcal{F}_{I,n}| = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{(n-k)!} = \sum_{k=0}^{n} k! \binom{n}{k}^2$$

(2)

which is sequence A002720 in the On-Line Encyclopedia of Integer Sequences.

Example 2. Let $f : \{1, 2, 4\} \rightarrow \{4\}$ be defined by $f(1) = 2$, $f(2) = 4$, and $f(4) = 1$. Clearly $f$ is injective. A directed labeled graph of this function is given by

```
1 -> 2 -> 4   3
```

and the associated matrix is

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Hence the result in (2) also counts the number of labeled directed graphs on $n$ vertices each of which has out-degree no greater than one and indegree no greater than one. Furthermore, the result in (2) counts the number of $n \times n$ binary matrices with each row sum and each column sum equal to 0 or to 1.

III. Acyclic Functions from Subsets of $[n]$ to $[n]$.

Let $\mathcal{F}_{A,n}$ denote the set $\{f : D \rightarrow [n] \mid D \text{ is a subset of } [n] \text{ and } f \text{ is acyclic}\}$. If subset $D$ has size $k$, then the number of acyclic functions from $D$ to $[n]$ is $(n-k)(n-k+k)^{k-1} = (n-k)n^{k-1}$. Summing over all subsets of $[n]$ we obtain

$$|\mathcal{F}_{A,n}| = \sum_{k=0}^{n} \binom{n}{k} (n-k)n^{k-1}$$

$$= \sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!k!} n^{k-1}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} n^k$$

$$= (n + 1)^{n-1}$$

(3)
Compare to OEIS sequence A000272(n) = \(n^{n-2}\), the number of trees on \(n\) labeled nodes.

**Example 3.** Let \(f : \{1, 2, 3\} \rightarrow \{4\}\) be defined by \(f(1) = 3\), \(f(2) = 4\), and \(f(3) = 4\). Here \(f\) contains no cycles. Also its labeled acyclic directed graph is

![Graph](image)

The associated binary matrix is

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Note. If the function has a cycle then the binary matrix or some deletion submatrix of it will be a permutation matrix. A deletion submatrix is a matrix obtained by deleting any number of row-column pairs.

For example, the function with the following matrix

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

has a cycle since deleting row 2 and column 2 results in the permutation matrix

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

We also note that \((n + 1)^{n-1}\) is the number of acyclic functions from \([n]\) to \([n + 1]\). \(\quad(4)\)
The one-to-one correspondence between the two sets of functions is illustrated by the following. Let \(f\) be an arbitrary function of \(F_{A,n}\) with domain \(D = \{d_1, \ldots, d_k\}\). Let \(g : [n] \rightarrow [n + 1]\) be defined by \(g(d_i) = f(d_i)\) for every \(d_i \in D\) and \(g(x) = n + 1\) for any \(x \notin D\). Clearly \(g\) is acyclic because every element of \([n]\) is sent by \(g\) or eventually sent under successive composition of \(g\) to the element \(n + 1\).

We can also obtain OEIS sequence A213326 by considering the acyclic functions from subsets of \([n + 1]\) of size \(n - 1\) or less mapped into \([n + 1]\). The number of acyclic functions from subsets of \([n + 1]\) mapped to \([n + 1]\) is \((n + 2)^n\) by (3). Now consider a size \(n\) subset, say \(D\), of \([n + 1]\). There are \(n + 1\) such subsets. The number of acyclic functions from \(D\) to \([n + 1]\) is \((n + 1)^{n-1}\) by (4). Hence the number of acyclic functions from subsets of size \(n - 1\) or less of \([n + 1]\) to \([n + 1]\) is given by

\[
(n + 2)^n - (n + 1)(n + 1)^{n-1} = (n + 2)^n - (n + 1)^n.
\] \(\quad(5)\)
IV. Acyclic and Injective Functions from Subsets of \([n]\) to \([n]\).

Let \(\mathcal{F}_{A,I,n} = \{ f : D \rightarrow [n] \text{ where } D \text{ is any subset of } [n], \ f \text{ is acyclic and injective} \} \). The cardinality \(c(n)\) of set \(\mathcal{F}_{A,I,n}\) is given by

\[
c(n) = \sum_{k=0}^{n-1} \binom{n}{k} \frac{(n-1)!}{(n-k-1)!}
\]

for \(n \geq 1\), which is identical to sequence A000262 in OEIS.

Similarly, \(c(n)\) is the number of labeled directed graphs on \(n\) vertices (i) that have no cycles, (ii) that have maximum outdegree 1, and (iii) that have maximum indegree 1. Equivalently, \(c(n)\) is the number of labeled-rooted skinny-tree forests on \(n\) nodes, where a skinny tree has no vertex with more than one child. Let \(k\) denote the number of trees. There are \(\binom{n}{k}\) ways to choose the roots, \(\binom{n-1}{k-1}\) ways to choose the number of descendants for each root, and \((n-k)!\) ways to permute those descendants. Summing over \(k\), we obtain

\[
c(n) = \sum_{k=1}^{n} \binom{n}{k} \frac{(n-1)}{(k-1)} (n - k)!
\]

which is equivalent to (6).

Example 4. The 13 labeled rooted skinny-tree forests on 3 vertices.

\[
\begin{array}{cccc}
1 & 2 & 3 \\
2 & 3 & 1 & 1 \\
1 & 3 & 1 & 2 & 2 & 3 & 3 & 2 & 2 & 1 & 3 & 1 \\
3 & 2 & 3 & 1 & 1 & 2 & 1 & 1 & 2 & 3 & 3 & 2 \\
2 & 3 & 1 & 1 & 2 & 3 & 3 & 2 & 2 & 1 & 3 & 1 \\
1 & 1 & 2 & 3 & 3 & 2 & 2 & 1 & 3 & 1
\end{array}
\]
MAPLE code for formula (6)

\[
b := n \to \sum \frac{\binom{n}{k} \cdot (n-1)!}{(n-k-1)!}, k = 0 \ldots n-1;
\]

\[
b := n \to \frac{\sum \binom{n}{k} \cdot (n-1)!}{(n-k-1)!}, k = 0
\]

> seq(b(n), n=1..20);

1, 3, 13, 73, 501, 4051, 37633, 394353, 4596553, 58941091, 824073141, 12470162233, 202976401213, 3535017524403, 65573803186921, 1290434218669921, 26846616451246353, 588633468315403843, 13564373693588558173, 32769792788608565441

V. Functions \( f \) from Subsets of \( \{1, 2, \ldots, n\} \) into \( \{1, 2, \ldots, n\} \) with \( f(x) \equiv 0 \pmod{x} \) for every \( x \) in the domain

\[
f(1) \in [n] \text{ or undefined when } 1 \notin \text{ domain } D
\]
\[
f(2) \in \{2, 4, \ldots\} \text{ or undefined , where } \{2, 4, \ldots\} \text{ has size } \lfloor n/2 \rfloor
\]
\[
f(3) \in \{3, 6, \ldots\} \text{ or undefined, where } \{2, 4, \ldots\} \text{ has size } \lfloor n/3 \rfloor
\]
\[
\vdots
\]
\[
f(n) \in \{n\} \text{ or undefined}
\]

Number of such functions = \( a(n) = \prod_{k=1}^{n} (\lfloor n/k \rfloor + 1) = A131385(n + 1) \)

Example. \( n = 3 \). \( a(n) = (3 + 1)(1 + 1)(1 + 1) = 16 \)

The 16 functions. Null function \( f : \emptyset \to \{1, 2, 3\} \)

\[
\begin{array}{|c|}
\hline
x & f(x) \\
\hline
1 & 1 \\
\hline
\end{array}
\]

(Vacuously holds)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 1 & x & 1 & x & 1 \\
\hline
1 & f(x) & 2 & f(x) & 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
x & 1 & x & 1 \\
\hline
1 & 2 & 2 & f(x) \\
\hline
2 & f(x) & 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|}
\hline
x & 1 & x & 1 \\
\hline
1 & 3 & 3 & f(x) \\
\hline
2 & f(x) & 3 \\
\hline
3 & f(x) & 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 1 & 2 & 3 & x & 1 & 2 \\
\hline
1 & 2 & 3 & f(x) & 2 & 2 \\
\hline
2 & f(x) & 3 & f(x) & 3 & 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 1 & 2 & 3 & x & 1 & 2 \\
\hline
1 & 2 & 3 & f(x) & 2 & 3 \\
\hline
2 & f(x) & 3 & 2 & f(x) & 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 1 & 2 & 3 & x & 1 & 2 \\
\hline
1 & 2 & 3 & f(x) & 2 & 3 \\
\hline
2 & f(x) & 3 & f(x) & 3 \\
\hline
3 & f(x) & 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 1 & 2 & 3 & x & 1 & 2 \\
\hline
1 & 2 & 3 & f(x) & 2 & 3 \\
\hline
2 & f(x) & 3 & f(x) & 3 \\
\hline
3 & f(x) & 3 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 1 & 2 & 3 & x & 1 & 2 \\
\hline
1 & 2 & 3 & f(x) & 2 & 3 \\
\hline
2 & f(x) & 3 & f(x) & 3 \\
\hline
3 & f(x) & 3 \\
\hline
\end{array}
\]
Maple Code

a := n -> product(floor(n/k) + 1, k = 1 .. n);

\[
b := n \rightarrow \prod_{k=1}^{n}(\floor{n/k} + 1)
\]

> simplify(a(n));

\[
b := n \rightarrow \prod_{k=1}^{n}(\floor{n/k} + 1)
\]

> seq(a(n), n = 1 .. 20);

2, 6, 16, 60, 144, 672, 1536, 6480, 19200, 76032, 165888, 1048320,
2257920, 8294400, 28311552, 126904320, 268738560, 1470873600,
3096576000, 16094453760