Assigning People into Labeled Groups with Leaders

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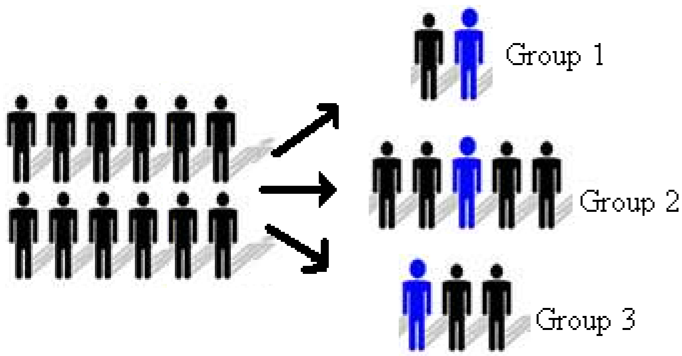


I. Counting Group Assignments

In how many ways can one form k numbered groups from n people and then choose a leader for each group? We assume the groups are nonempty and not necessarily of the same size. We will let $T_{n,k}$ denote the number of ways. Since there are $\frac{n!}{(n-k)!}$ ways to assign leaders to the k numbered groups and there are k^{n-k} ways to map the remaining $n-k$ people to the k groups, we obtain

$$T_{n,k} = \frac{n!}{(n-k)!}\, k^{n-k} \text{ for } k = 1, ..., n. \tag{1}$$

We note that $T_{n,1} = n$ since there are n choices for the leader of the single group. Also, $T_{n,n} = n!$ since, in this case, each of the n groups consist solely of a leader and there are $n!$ ways to assign the n people to the n labeled groups.

Example 1. $T_{3,2} = 12$ since there are 12 ways to form group 1 and group 2, both with leaders, using people $p_1,\ p_2,$ and p_3. A leader will be designated L_j if person p_j is assigned as the leader of the group. The 12 possible assignments are given below.

Assignment	Group1	Group2
1	$L_1,\ p_2$	L_3
2	$L_1,\ p_3$	L_2
3	L_1	L_2, p_3
4	L_1	p_2, L_3
5	$L_2, p1$	L_3
6	$L_2,\ p_3$	L_1
7	L_2	$L_1,\ p_3$
8	L_2	$p_1,\ L_3$
9	$L_3,\ p_2$	L_1
10	$L_3,\ p_1$	L_2
11	L_3	$L_1,\ p_2$
12	L_3	$p_1,\ L_2$

◇

Below is a partial table for values of $T_{n,k}$.

Table for $T_{n,k}$

n\k	1	2	3	4	5	6	7	8	9
1	1								
2	2	2							
3	3	12	6						
4	4	48	72	24					
5	5	160	540	480	120				
6	6	480	3240	5760	3600	720			
7	7	1344	17010	53760	6300	30240	5040		
8	8	3584	81648	430080	840000	725760	282240	40320	
9	9	9216	367416	3096576	9450000	13063680	8890560	2903040	362880

Exponential Generating Functions for $T_{n,k}$

Let g_k denote the (exponential) generating function for sequence $T_{n,k}$. Then we obtain

$$g_1(x) = \sum_{n=1}^{\infty} (T_{n,1}) x^n/n! = \sum_{n=1}^{\infty} n\, x^n/n! = x\sum_{n=1}^{\infty} x^{n-1}/(n-1)! = xe^x,$$

and

$$g_k(x) = \sum_{n=1}^{\infty} \tfrac{n!}{(n-k)!} k^{n-k} x^n/n! = x^k \sum_{n=k}^{\infty} (kx)^{n-k}/(n-k)! = x^k e^{kx} = (xe^x)^k.$$

A Binomial Identity for $T_{n,k}$

The generating functions above imply that $g_k(x) = (xe^x)^k = (g_1(x))^k$. This suggests that $T_{n,k}$ (viewed as a function of nonnegative integers n and k) will satisfy the binomial identity given in the theorem below.

Theorem. Let k_1 and k_2 be positive integers with $k_1 + k_2 = k$. Then

$$T_{n,k} = \sum_{j=0}^{n} \binom{n}{j} T_{j,k_1} T_{n-j,k_2}.$$

Proof.

$$\sum_{j=0}^{n} \binom{n}{j} T_{j,k_1} T_{n-j,k_2} = \sum_{j=0}^{n} \tfrac{n!}{(n-j)!j!} \tfrac{j!}{(j-k_1)!} k_1^{j-k_1} \tfrac{(n-j)!}{(n-j-k_2)!} k_2^{n-j-k_2}$$

$$= \tfrac{n!}{(n-k_1-k_2)!} \sum_{j=k_1}^{n-k_2} \tfrac{(n-k_1-k_2)!}{(n-j-k_2)!(j-k_1)!} k_1^{j-k_1} k_2^{n-j-k_2}$$

Letting $r = j - k_1$, we obtain

$$\sum_{j=0}^{n} \binom{n}{j} T_{j,k_1} T_{n-j,k_2} = \tfrac{n!}{(n-k_1-k_2)!} \sum_{r=0}^{n-k_1-k_2} \binom{n-k_1-k_2}{r} k_1^r k_2^{n-k_1-k_2-r}$$

$$= \tfrac{n!}{(n-k_1-k_2)!} (k_1 + k_2)^{n-k_1-k_2}$$

$$= \tfrac{n!}{(n-k)!} k^{n-k} \qquad \square$$

Alternative Descriptions for $T_{n,k}$

(a) Counting functions.

Consider a function $f:\{1,2,...,n\}\to\{1,2,...,2k\}$ such that Card$(f^{-1}(i))=1$ for $i=1,...,k$. The set of all such functions has cardinality $\frac{n!}{(n-k)!}k^{(n-k)}$. This is readily seen since there are $\frac{n!}{(n-k)!}$ ways to choose the k elements of the domain and map them injectively to $\{1,...,k\}$ and there are k^{n-k} ways to map the remaining $n-k$ elements of the domain to $\{k+1,...,2k\}$.

Example 2. For example, let $f:\{1,2,3\}\to\{1,2,3,4\}$ be defined by

x	1	2	3
$f(x)$	4	1	2

.

Here $n=3$ and $k=2$. Note that the pre-image of 1 and the pre-image of 2 are both singleton sets, satisfying the condition that Card$(f^{-1}(i))=1$ for $i=1,...,k$.
We note that the function induces an ordered partition of the domain into 2 sets each with a "designated" element, that is $\{1,2,3\}=\{\mathbf{2}\}\cup\{\mathbf{3},1\}$. The designated elements are the pre-images of 1 and 2. The number 1 is an element of the second set above since $f(1)=4=2$ (mod k).

On the other hand, let $f:\{1,2,3\}\to\{1,2,3,4\}$ be defined by

x	1	2	3
$f(x)$	3	2	1

.

The induced ordered partition here is given by $\{1,2,3\}=\{\mathbf{3},1\}\cup\{\mathbf{2}\}$ $\diamond$

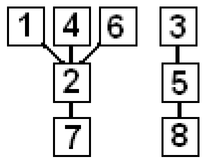
(b) Counting forests.

$T_{n,k}$ is the number of labeled forests on $n+k$ vertices with

(i) exactly k rooted trees labeled $n+1,...,n+k$,
(iii) each tree of height at most 2, and
(ii) each root having exactly one child.

To see this, note that there are $\frac{n!}{(n-k)!}$ ways to assign the k child vertices of the k roots and there are k^{n-k} to assign the remaining $n-k$ vertices for the second generation.

Example 3. With $n = 6$ and $k = 2$, an example of a forest satisfying the three conditions above is shown below.



Integer Sequences for Several $T_{n.k}$

As previously noted, $T_{n,1} = n$ and $T_{n,n} = n!$, both well known sequences. In particular, $T_{n,1}$ is sequence A000027 and $T_{n,n}$ is sequence A000142 in the *On-Line Encyclopedia of Integer Sequences* (OEIS).

Also, sequence $T_{n,2}$ (0, 2, 12, 48, 160, 480, 1344, ...) is listed in the *On-Line Encyclopedia of Integer Sequences* as sequence A001815, which has the description "a(n) is the number of ways to assign n distinct contestants to two (not necessarily equal) distinct teams and then choose a captain for each team. [From Geoffrey Critzer (critzer.geoffrey(AT)usd443.org), Apr 07 2009]" .

Sequence $T_{n,3}$ is sequence A052791 in OEIS, described there as "the number of surjective functions f:{1,2,...,n}->{1,2,3} with a designated pre-image of 1, 2, and 3."

II. Counting Group Assignments with Number of Groups Unspecified

In how many ways can n people form (any number of) nonempty labeled groups, each with a designated leader? Letting T_n denote the number of ways, we have

$$T_n = \sum_{k=1}^{n} T_{n,k}$$

$$= n! \sum_{k=1}^{n} \frac{k^{n-k}}{(n-k)!}.$$

Sequence T_n

$T_n\,(n = 1...20)$: 1, 4, 21, 148, 1305, 13806, 170401, 2403640, 38143377, 672552730, 13044463641, 276003553860, 6326524990825, 156171026562838, 4130464801497105, 116526877671782896, 3492868475952497313, 110856698175372359346, 3713836169709782989993, 130966414749485504586940, ...
(See sequence A006153 at http://oeis.org/A006153.)

Exponential Generating Function for T_n

Let g denote the exponential generating function for T_n defined by $g(x) = \sum_{n=1}^{\infty} T_n \frac{x^n}{n!}$. Thus we have

$$\begin{aligned}
g(x) &= \sum_{n=1}^{\infty} n! \sum_{k=1}^{n} \frac{k^{n-k}}{(n-k)!} \frac{x^n}{n!} \\
&= \sum_{k=1}^{\infty} x^k \sum_{n=k}^{\infty} \frac{k^{n-k}}{(n-k)!} x^{n-k} \\
&= \sum_{k=1}^{\infty} x^k \sum_{j=0}^{\infty} \frac{k^j}{j!} x^j \\
&= \sum_{k=1}^{\infty} x^k \, e^{xk} \\
&= \frac{1}{1-xe^x} - 1 \\
&= \frac{xe^x}{1-xe^x}.
\end{aligned}$$