I. Counting Group Assignments

In how many ways can one form \( k \) numbered groups from \( n \) people and then choose a leader for each group? We assume the groups are nonempty and not necessarily of the same size. We will let \( T_{n,k} \) denote the number of ways. Since there are \( \frac{n!}{(n-k)!} \) ways to assign leaders to the \( k \) numbered groups and there are \( k^{n-k} \) ways to map the remaining \( n - k \) people to the \( k \) groups, we obtain

\[
T_{n,k} = \frac{n!}{(n-k)!} \ k^{n-k} \quad \text{for} \quad k = 1, \ldots, n. \tag{1}
\]

We note that \( T_{n,1} = n \) since there are \( n \) choices for the leader of the single group. Also, \( T_{n,n} = n! \) since, in this case, each of the \( n \) groups consist solely of a leader and there are \( n! \) ways to assign the \( n \) people to the \( n \) labeled groups.
**Example 1.** \( T_{3,2} = 12 \) since there are 12 ways to form group 1 and group 2, both with leaders, using people \( p_1, p_2, \) and \( p_3. \) A leader will be designated \( L_j \) if person \( p_j \) is assigned as the leader of the group. The 12 possible assignments are given below.

<table>
<thead>
<tr>
<th>Assignment</th>
<th>Group1</th>
<th>Group2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( L_1, p_2 )</td>
<td>( L_3 )</td>
</tr>
<tr>
<td>2</td>
<td>( L_1, p_3 )</td>
<td>( L_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( L_1 )</td>
<td>( L_2, p_3 )</td>
</tr>
<tr>
<td>4</td>
<td>( L_1 )</td>
<td>( p_2, L_3 )</td>
</tr>
<tr>
<td>5</td>
<td>( L_2, p_1 )</td>
<td>( L_3 )</td>
</tr>
<tr>
<td>6</td>
<td>( L_2, p_3 )</td>
<td>( L_1 )</td>
</tr>
<tr>
<td>7</td>
<td>( L_2 )</td>
<td>( L_1, p_3 )</td>
</tr>
<tr>
<td>8</td>
<td>( L_2 )</td>
<td>( p_1, L_3 )</td>
</tr>
<tr>
<td>9</td>
<td>( L_3, p_2 )</td>
<td>( L_1 )</td>
</tr>
<tr>
<td>10</td>
<td>( L_3, p_1 )</td>
<td>( L_2 )</td>
</tr>
<tr>
<td>11</td>
<td>( L_3 )</td>
<td>( L_1, p_2 )</td>
</tr>
<tr>
<td>12</td>
<td>( L_3 )</td>
<td>( p_1, L_2 )</td>
</tr>
</tbody>
</table>

Below is a partial table for values of \( T_{n,k}. \)

<table>
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<tr>
<th>( n )</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<td>2</td>
<td>2</td>
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<td>6300</td>
<td>30240</td>
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<td>8890560</td>
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</tr>
</tbody>
</table>
Exponential Generating Functions for $T_{n,k}$

Let $g_k$ denote the (exponential) generating function for sequence $T_{n,k}$. Then we obtain

$$g_1(x) = \sum_{n=1}^{\infty} (T_{n,1})x^n/n! = \sum_{n=1}^{\infty} n x^n/n! = x \sum_{n=1}^{\infty} x^{n-1}/(n-1)! = xe^x,$$

and

$$g_k(x) = \sum_{n=1}^{\infty} \frac{n!}{(n-k)!} k^{n-k} x^n/n! = x^k \sum_{n=k}^{\infty} (kx)^{n-k}/(n-k)! = x^k e^{kx} = (xe^x)^k.$$

A Binomial Identity for $T_{n,k}$

The generating functions above imply that $g_k(x) = (xe^x)^k = (g_1(x))^k$. This suggests that $T_{n,k}$ (viewed as a function of nonnegative integers $n$ and $k$) will satisfy the binomial identity given in the theorem below.

**Theorem.** Let $k_1$ and $k_2$ be positive integers with $k_1 + k_2 = k$. Then

$$T_{n,k} = \sum_{j=0}^{n} \binom{n}{j} T_{j,k_1} T_{n-j,k_2}.$$

**Proof.**

$$\sum_{j=0}^{n} \binom{n}{j} T_{j,k_1} T_{n-j,k_2} = \sum_{j=0}^{n} \frac{n!}{(n-j)!j!(j-k_1)!} j^{k_1} (n-j)! (n-j-k_2)! k_2^{n-j-k_2}$$

$$= \frac{n!}{(n-k_1-k_2)!} \sum_{j=k_1}^{n-k_2} \frac{(n-k_1-k_2)!}{(n-j-k_2)!(j-k_1)!} j^{k_1} k_2^{n-j-k_2}$$

Letting $r = j - k_1$, we obtain

$$\sum_{j=0}^{n} \binom{n}{j} T_{j,k_1} T_{n-j,k_2} = \frac{n!}{(n-k_1-k_2)!} \sum_{r=0}^{n-k_2} \binom{n-k_1-k_2}{r} k_1^r k_2^{n-k_1-k_2-r}$$

$$= \frac{n!}{(n-k_1-k_2)!} (k_1 + k_2)^{n-k_1-k_2}$$

$$= \frac{n!}{(n-k)!} k^{n-k} \quad \square$$
Alternative Descriptions for $T_{n,k}$

(a) Counting functions.

Consider a function $f : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, 2k\}$ such that $\text{Card}(f^{-1}(i)) = 1$ for $i = 1, \ldots, k$. The set of all such functions has cardinality $\frac{n!}{(n-k)!}k^{(n-k)}$. This is readily seen since there are $\frac{n!}{(n-k)!}$ ways to choose the $k$ elements of the domain and map them injectively to $\{1, \ldots, k\}$ and there are $k^{n-k}$ ways to map the remaining $n-k$ elements of the domain to $\{k+1, \ldots, 2k\}$.

Example 2. For example, let $f : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$ be defined by

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Here $n = 3$ and $k = 2$. Note that the pre-image of 1 and the pre-image of 2 are both singleton sets, satisfying the condition that $\text{Card}(f^{-1}(i)) = 1$ for $i = 1, \ldots, k$.

We note that the function induces an ordered partition of the domain into 2 sets each with a “designated” element, that is $\{1, 2, 3\} = \{2\} \cup \{3, 1\}$. The designated elements are the pre-images of 1 and 2. The number 1 is an element of the second set above since $f(1) = 4 = 2 \mod k$.

On the other hand, let $f : \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$ be defined by

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

The induced ordered partition here is given by $\{1, 2, 3\} = \{3, 1\} \cup \{2\}$

(b) Counting forests.

$T_{n,k}$ is the number of labeled forests on $n + k$ vertices with

(i) exactly $k$ rooted trees labeled $n + 1, \ldots, n + k$,

(ii) each root having exactly one child,

(iii) each tree of height at most 2.

To see this, note that there are $\frac{n!}{(n-k)!}$ ways to assign the $k$ child vertices of the $k$ roots and there are $k^{n-k}$ to assign the remaining $n - k$ vertices for the second generation.
Example 3. With \( n = 6 \) and \( k = 2 \), an example of a forest satisfying the three conditions above is shown below.

![Diagram of a forest with labeled nodes and edges]

**Integer Sequences for Several \( T_{n,k} \)**

As previously noted, \( T_{n,1} = n \) and \( T_{n,n} = n! \), both well known sequences. In particular, \( T_{n,1} \) is sequence A000027 and \( T_{n,n} \) is sequence A000142 in the On-Line Encyclopedia of Integer Sequences (OEIS).

Also, sequence \( T_{n,2} \) (0, 2, 12, 48, 160, 480, 1344, ...) is listed in the On-Line Encyclopedia of Integer Sequences as sequence A001815, which has the description "a(n) is the number of ways to assign n distinct contestants to two (not necessarily equal) distinct teams and then choose a captain for each team. [From Geoffrey Critzer (critzer.geoffrey(AT)usd443.org), Apr 07 2009]".

Sequence \( T_{n,3} \) is sequence A052791 in OEIS, described there as “the number of surjective functions f:{1,2,...,n}->{1,2,3} with a designated pre-image of 1, 2, and 3.”

**II. Counting Group Assignments with Number of Groups Unspecified**

In how many ways can \( n \) people form (any number of) nonempty labeled groups, each with a designated leader? Letting \( T_n \) denote the number of ways, we have

\[
T_n = \sum_{k=1}^{n} T_{n,k}
\]

\[
= n! \sum_{k=1}^{n} \frac{n-k}{(n-k)!}.
\]

**Sequence \( T_n \)**

\( T_n \ (n = 1...20) \): 1, 4, 21, 148, 1305, 13806, 170401, 2403640, 38143377, 672552730, 13044463641, 276003553860, 6326524990825, 156171026562838, 4130464801497105, 116526877671782896, 3492868475952497313, 110856698175372359346, 3713836169709782989993, 130966414749485504586940, ...

(See sequence A006153 at http://oeis.org/A006153.)
Exponential Generating Function for $T_n$

Let $g$ denote the exponential generating function for $T_n$ defined by $g(x) = \sum_{n=1}^{\infty} T_n \frac{x^n}{n!}$. Thus we have

\[
g(x) = \sum_{n=1}^{\infty} n! \sum_{k=1}^{n} \frac{k^{n-k}}{(n-k)!} \frac{x^n}{n!}
\]

\[=
\sum_{k=1}^{\infty} x^k \sum_{n=k}^{\infty} \frac{k^{n-k}}{(n-k)!} \frac{x^n}{n!}
\]

\[=
\sum_{k=1}^{\infty} x^k \sum_{j=0}^{\infty} \frac{k^j}{j!} x^j
\]

\[=
\sum_{k=1}^{\infty} x^k e^{xk}
\]

\[=
\frac{1}{1-xe^x} - 1
\]

\[=
\frac{xe^x}{1-xe^x}.
\]