Counting $n \times 2$ Simple Rectangular Mazes

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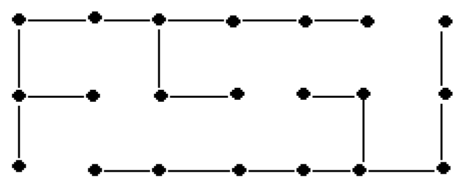
Definition. An n x m simple rectangular maze (SRM) is a graph G with vertex set $\{0,1,...,n\} \times \{0,1,...,m\}$ that satisfies the following two properties:

(i) G consists of two orthogonal trees;
(ii) one tree has a path that sequentially connects $(0,0), (0,1), ...(0,m), (1,m), ..., (n-1,m)$, and the other tree has a path that sequentially connects $(1,0), (2,0), ..., (n,0), (n,1), ..., (n,m)$.

Before counting the number of $(n \times 2)$ SRM's, we illustrate several properties with an example.

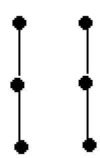
Example. Here is a simple 6×2 rectangular maze.



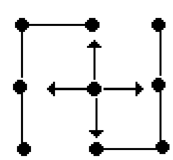
The edges of a SRM can be considered directed edges. Without loss of generality, we assume the direction of an edge from a vertex to be in the direction of the path from that vertex to an exit vertex. Under this assumption, the out-degree of all the vertices is one. A vertex of a maze can be classified as either external or internal. Viewing a maze as a house, the external vertices are those vertices that lie on the outside walls of the house, and the internal vertices lie inside the house.

Since the external vertices and edges are fixed, a SRM is completely determined by the internal vertices and the directions of their out-edges. For example, the SRM above can be described by the ordered 5-tuple (W, N, W, S, E), where W denotes west, N denotes north, etc.

Counting. Let $c(n)$ denote the number of ($n \times 2$) SRM's. For $n = 1$, there is one 1×2 SRM. Thus $c(1) = 1$.

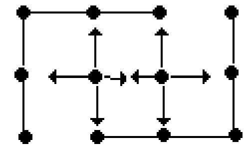


For $n = 2$, there are four 2×2 $SRM's$ because there is one internal vertex and their are 4 possible directions $(N, E, S,$ or $W)$ for the edge emanating from it.



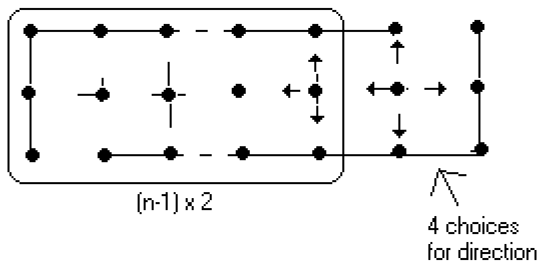
Hence $c(2) = 4$.

For $n = 3,$ consider the following maze diagram ...

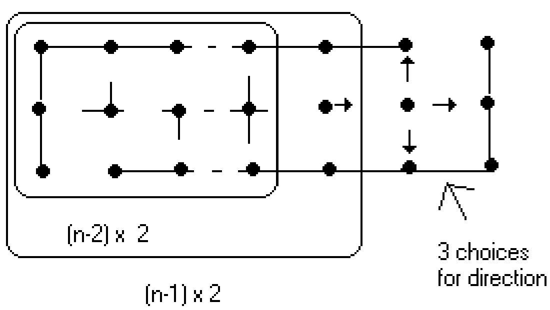


There are 4 possible directions for the left internal edge and 4 directions for the right internal edge. However, since the maze determined by the pair $(E, W),$ that is (• $\rightarrow$, $\leftarrow$ •), leads to 3 trees in the graph, it is not a SRM and must be excluded. Therefore, $c(3) = 4(4) - 1 = 15.$

For $n \geq 4,$ we use the same method as used in the $n = 3$ case above to count $c(n).$ As an another internal vertex is added to the right side of the maze, each of the $(n-1) \times 2$ mazes generates either 4 or 3 new $(n \times 2)$ mazes. The additional vertex has 4 possible directions for its edge, except



or



when direction W is not allowed, which occurs when the rightmost vertex of the $(n-1) \times 2$ maze has edge direction E. The number of $(n-1) \times 2$ mazes that has edge direction E for the rightmost vertex is $c(n-2)$ since each of the $(n-2) \times 2$ mazes gives rise to a $(n-1) \times 2$ mazes that has edge direction E for the rightmost vertex. Therefore,

$$c(n) = 4(c(n-1) - c(n-2)) + 3c(n-2)$$

$$= 4c(n-1) - c(n-2). \quad (1)$$

Since $c(0)$ is trivially 0, the recursive count in (1) holds for $n \geq 2.$

Closed form. To find a closed form for the linear recursive sequence defined by $c(n)$ in (1), we note that $c(n+2) - 4c(n+1) + c(n) = 0,$ which implies that the characteristic polynomial f for the sequence is given by $f(x) = x^2 - 4x + 1.$

Hence, for all n, $c(n)$ satisfies

$$c(n) = k_1 z_1^n + k_2 z_2^n \tag{2}$$

where z_1 and z_2 are the zeros of f. Solving $x^2 - 4x + 1 = 0$ gives us $z_1 = 2 - \sqrt{3}$ and $z_2 = 2 + \sqrt{3}$. To find k_1 and k_2, note that, when $n = 0$, equation (2) implies $k_1 = -k_2$. Furthermore, when $n = 1$, we get $c(1) = 1 = k_1(2 - \sqrt{3}) + (-k_1)(2 + \sqrt{3})$. Thus $k_1 = -\frac{1}{2\sqrt{3}}$ and $k_2 = \frac{1}{2\sqrt{3}}$. Therefore, $c(n)$, the number of $2 \times n$ simple rectangular mazes is given by

$$c(n) = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}} \tag{3}$$

for $n = 0, 1, 2 \ldots$.