Notes on Finite Monotonic and Non-monotonic Functions

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A monotonic function is a function which is either non-decreasing or non-increasing over its domain. We consider monotonic functions whose domain and codomain are finite sets of integers and then derive the cardinalities of sets of such functions. We will let $[m]$ denote the set of the first m positive integers, that is, $[m] = \{1, 2, ..., m\}$; also, we will use $[0]$ to denote the empty set.

I. Strictly Increasing (Decreasing) Functions $f : [k] \to [n]$

A function f is strictly increasing if $z > y$ implies $f(z) > f(y)$.

An example of a finite increasing function $f{:}\{1, .., 4\} \to \{1, ..., 6\}$ is given in the following table

x	1	2	3	4
$f(x)$	2	3	5	6

.

For $0 \leq k \leq n$, let $F = \{f : [k] \to [n]$ such that f is strictly increasing$\}$. The cardinality c of F is given by $c(n, k) = \binom{n}{k}$ since there are $\binom{n}{k}$ ways to choose the k distinct ordered elements $f(1), ..., f(n)$ from the codomain $\{1, ..., n\}$. For example, there are $\binom{4}{2} = 6$ increasing functions from [2] to [4], namely, $\{(1, 1), (2, 2)\}$, $\{(1, 1), (2, 3)\}$, $\{(1, 1), (2, 4)\}$, $\{(1, 2), (2, 3)\}$, $\{(1, 2), (2, 4)\}$, and $\{(1, 3), (3, 4)\}$.

Based on symmetry, the number of strictly decreasing functions from $[k]$ to $[n]$ is the same as the number of strictly increasing functions. Also, for the case when $k = n = 0$, we include the null function since it vacuously satisfies the definition of a strictly increasing function.

II. Non-decreasing functions $f : [k] \to [n]$

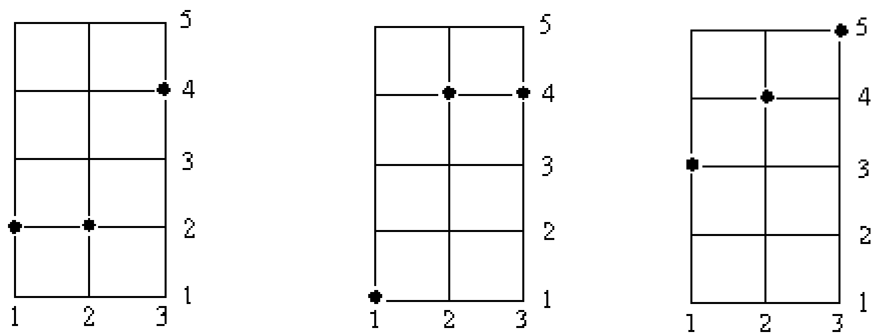
We now consider finite non-decreasing functions. A function f is non-decreasing if $f(z) \geq f(x)$ whenever $z > x$.

Example 1. An example of a non-decreasing function $f\colon[4]\to[6]$ is given in the table below.

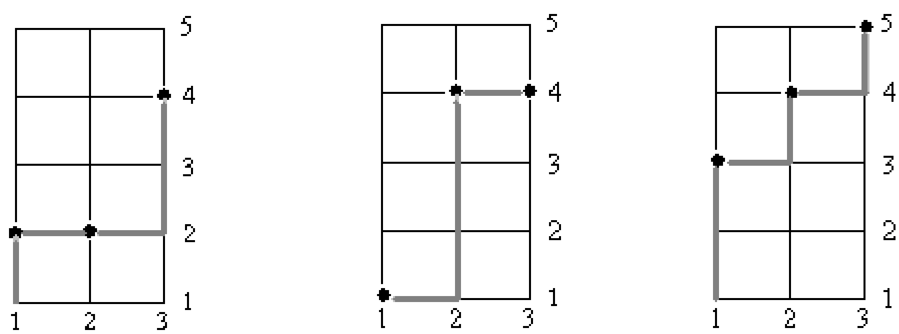
x	1	2	3	4
$f(x)$	2	3	3	6

. ◇

Example 2. Consider non-decreasing functions $f:[3]\to[5]$. Three of this type are shown in the graphs below.



Each function corresponds to a unique direct walk from $(1,1)$ to $(3,f(3))$ on the integer lattice of the first quadrant.



Conversely, each direct walk from $(1,1)$ to $(3,f(3))$ corresponds to a unique function. If we count the number of walks as $f(3)$ runs from 1 through 5, we obtain $\sum_{j=1}^{5}\binom{j+1}{2}$ since there are $j+1$ steps to each walk and we must choose 2 steps in which we go to the right. Hence we get $\sum_{j=1}^{5}\binom{j+1}{2}=\binom{2}{2}+\binom{3}{2}+\ldots+\binom{6}{2}=\binom{7}{3}=35$ (using the "hockey-stick" formula for binomial coefficients). ◇

Cardinalities

We could derive the cardinality c of the set $\{f : [k] \to [n] \mid f$ is non-decreasing$\}$ by counting direct walks on a rectangular grid as done in example 2 above, but we will take a different route that is given below.

Consider all arrangements of $n-1$ slashes "/" and the k symbols "f". There are $\binom{n-1+k}{k}$ such arrangements.

For example, if $n = 6$ and $k = 4$, one such arrangement is

$$/f//fff//.$$

The particular arrangement above translates to the non-decreasing function f given by

x	1	2	3	4
$f(x)$	2	4	4	4

Above, $f(1) = 2$ since the first f appears between the first and second slashes, $f(2) = f(3) = f(4) = 4$ since the second, third, and fourth f's all appear between the third and fourth slashes.

Another arrangement is $f//f/f//f$ which translates to the function f given by

x	1	2	3	4
$f(x)$	1	3	4	6

Above, $f(1) = 1$ since the first f appears prior to the first slash, $f(2) = 3$ since the second f appears between the second and third slashes, $f(3) = 4$ since the third f appears between the third and fourth slashes, and $f(4) = 6$ since the fourth f appears following the fifth slash.

Conversely, the function f given by

x	1	2	3	4
$f(x)$	2	3	5	6

corresponds uniquely to the arrangement

$$/f/f//f/f,$$

and the function f given by

x	1	2	3	4
$f(x)$	1	1	6	6

corresponds uniquely to the arrangement

$$ff/////ff.$$

As illustrated above, there is a one-to-one correspondence between the set of all distinguishable arrangements of $n-1$ slashes and k "f" symbols and the set of all non-decreasing functions from $[k]$ to $[n]$. Thus we have the following theorem

Theorem. The number of non-decreasing functions $f:[k]\rightarrow[n]$ is $\binom{n+k-1}{k}$.

Partial Table of $\binom{n+k-1}{k}$

$n\backslash k$	0	1	2	3	4	5	6
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	10	15	21	28
4	1	4	10	20	35	56	84
5	1	5	15	35	70	126	210
6	1	6	21	56	126	252	462
7	1	7	28	84	210	462	924

Note that the table is a left-tilted version of Pascal's triangle.

We also note that integer sequence A059481 in the *Online Encyclopedia of Integer Sequences* (http://oeis.org/A059481) is given by the triangular array $T(n,k)=\binom{n+k-1}{k}$, $0\le k\le n$.

Generating Function for Cardinalities

Let G_k denote the generating function for $\binom{n+k-1}{k}$ with k fixed. We derive G as follows:

$$G_k(x)=\sum_{n=0}^{\infty}\binom{n+k-1}{k}x^n=x\sum_{n=1}^{\infty}\binom{n+k-1}{k}x^{n-1}=x\sum_{n=1}^{\infty}\frac{(n+k-1)!}{k!}\frac{x^{n-1}}{(n-1)!}$$

$$=x\sum_{m=0}^{\infty}\frac{(m+k)!}{k!}\frac{x^m}{m!}$$

$$=\frac{x}{(1-x)^k}.$$

Example. The series for $G_3(x)=\frac{x}{(1-x)^4}$ is $x+4x^2+10x^3+20x^4+35x^5+56x^6+84x^7+O(x^8)$.

III Non-monotonic functions f : $[k]\rightarrow[m]$

Consider all non-monotonic functions f:$[k]\rightarrow[m]$ where integers j and m are greater than 1. There are $m^k - 2\binom{m+k-1}{k} + m$ such functions since there are m^k functions of all types from $[k]\rightarrow[m]$, $\binom{m+k-1}{k}$ non-decreasing functions, $\binom{m+k-1}{k}$ non-increasing functions, and m functions that are both non-increasing and non-decreasing (i.e., constant functions). Thus, letting $c(m,k)$ denote the cardinality of this set of functions, we have $c(m,k) = m^k - 2\binom{m+k-1}{k} + m$

Using *Maple*, [>seq(seq(m^k-2*binomial(m+k-1,k)+m,k=1..10),m=1..10);] we obtain the following partial table for $c(m,k)$.

	1	2	3	4	5	6	7	8	9	10
1	0,	0,	0,	0,	0,	0,	0,	0,	0,	0,
2	0,	0,	2,	8,	22,	52,	114,	240,	494,	1004,
3	0,	0,	10,	54,	204,	676,	2118,	6474,	19576,	58920,
4	0,	0,	28,	190,	916,	3932,	16148,	65210,	261708,	1048008,
5	0,	0,	60,	490,	2878,	15210,	77470,	389640,	1951700,	9763628,
6	0,	0,	110,	1050,	7278,	45738,	278358,	1677048,	10073698,	60460176,
7	0,	0,	182,	1988,	15890,	115808,	820118,	5758802,	40343604,	282459240,
8	0,	0,	280,	3444,	31192,	258720,	2090296,	16764354,	134194856,	1073702936,
9	0,	0,	408,	5580,	56484,	525444,	4770108,	43020990,	387371878,	3486696894,
10	0,	0,	570,	8580,	96006,	990000,	9977130,	99951390,	999902770,	9999815254

We can create a triangle of nonzero values if we let $n = m + k$ with $3 \leq k \leq n-2$. We obtain

$$T(n,k) = \text{number of non-monotonic functions from } [k] \text{ to } [n-k], \\ = (n-k)^k - 2\binom{n-1}{k} + (n-k)$$

$n\backslash k$	3	4	5	6	7		
5	2						
6	10	8					
7	28	54	22				
8	60	190	204	52			
9	110	490	916	676	114		
10	182	1050	2878	3932	2118	240	
11	280	1988	7278	15210	16148	6474	494

Example $T(6,4) = 8$ since there are 8 non-monotonic functions from [4] to [2], namely, <$f(1), f(2), f(3), f(4)$>. given by <1,1,2,1>, <1,2,1,1>, <1,2,2,1>, <1,2,1,2>, <2,2,1,2>, <2,1,2,2>, <2,1,1,2>, and <2,1,2,1>.