Notes on Subsets of $\{1, 2, ..., n\}$ that Contain No Consecutive Integers

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Let $[n]$ denote the set of the first n positive integers, that is, $[n] = \{1, ..., n\}$. There are 2^n subsets of $[n]$. How many of these subsets contain no consecutive integers? In particular, how many size-k subsets contain no consecutive integers? Let $c(n, k)$ denote the number of size-k subsets that contain no consecutive integers.

Example 1. (i) If $n = 8$ and $k = 2$, the subsets are

$$
\begin{array}{rrrrrr}
\{1,3\}, & \{1,4\}, & \{1,5\}, & \{1,6\}, & \{1,7\}, & \{1,8\} \\
 & \{2,4\}, & \{2,5\}, & \{2,6\}, & \{2,7\}, & \{2,8\} \\
 & & \{3,5\}, & \{3,6\}, & \{3,7\}, & \{3,8\} \\
 & & & \{4,6\}, & \{4,7\}, & \{4,8\} \\
 & & & & \{5,7\}, & \{5,8\} \\
 & & & & & \{6,8\}
\end{array}
$$

giving us a count $c(8, 2) = 1 + 2 + ... + 6 = \binom{7}{2} = 21$.

(ii) If $n = 8$ and $k = 3$, the subsets are

$$
\begin{array}{l}
\{1,3,5\},\ \{1,3,6\},\ \{1,3,7\},\ \{1,3,8\} \\
\{1,4,6\},\ \{1,4,7\},\ \{1,4,8\} \\
\{1,5,7\}.\ \{1,5,8\} \\
\{1,6,8\} \\
\{2,4,6\},\ \{2,4,7\},\ \{2,4,8\} \\
\{2,5,7\},\ \{2,5,8\} \\
\{2,6,8\} \\
\{3,5,7\},\ \{3,5,8\} \\
\{3,6,8\} \\
\{4,6,8\}.
\end{array}
$$

There are $c(8, 3) = \binom{5}{2}+\binom{4}{2}+\binom{3}{2}+\binom{2}{2} = \binom{6}{3} = 20$ subsets.

(iii). If $n = 8$ and $k = 4$ the subsets are

$$
\begin{array}{l}
\{1,3,5,7\},\ \{1,3,5,8\},\ \{1,3,6,8\},\ \{1,4,6,8\} \\
\{2,4,6,8\}
\end{array}
$$

There are $c(8, 4) = \binom{5}{4} = 5$ subsets. ◇

I. Size-2 Subsets that Contain No Consecutive Integers

We first look at the case of size-2 subsets. For example, there are 6 size-2 subsets of $\{1,2,3,4,5\}$, namely, $\{1,3\},\ \{1,4\},\ \{1,5\},\ \{2,4\},\ \{2,5\},$ and $\{3,5\}$. Hence $c(5,2)=6$. To find a general formula for $c(n,2)$, we use the principle of inclusion/exclusion to obtain

$$\begin{aligned} c(n,2) &= \text{the number of all size-2 subsets} \\ &\quad - \text{the number of subsets that contain } i \text{ and } i+1 \text{ where } i \in [n-1] \\ &= \binom{n}{2} - (n-1) \\ &= \binom{n-1}{2}. \end{aligned} \tag{1}$$

An alternative approach to formula (1) uses the number of size-2 subsets that contain no consecutive integers and that has minimum element m which we denote by $c(n,2,m)$. Clearly $c(n,2,m)=n-m-1$ since the subsets being counted are $\{m,m+2\},\{m,m+3\},...,\{m,m+n-m\}$. Therefore,

$$\begin{aligned} c(n,2) &= \sum_{m=1}^{n-2} c(n,2,m) \\ &= \sum_{m=1}^{n-2} (n-m-1) \\ &= \sum_{j=1}^{n-2} j \quad \text{(upon reversing the order of summation)} \\ &= \binom{n-1}{2}. \end{aligned} \tag{2}$$

[See OEIS integer sequence A161680, at http://oeis.org/A161680, which sequence is given by $a(n)=\binom{n}{2}$, and $\binom{n}{2}$ = number of size-2 subsets of $\{0,1,...,n\}$ that contain no consecutive integers.]

Initial Values for $c(n,2)$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$c(n,2)$	0	0	0	1	3	6	10	15	21	28	36	45

II. Derivation of A General Formula for $c(n, k)$, the Number of Size-k Subsets of $[n]$ that Contain No Consecutive Integers

Consider a string of symbols consisting of k A's and $(n+1-2k)$ B's. There are $\binom{n+1-k}{k}$ ways to shuffle the symbols in the string. For each resulting string, replace each B symbol in the string with the number 0. If $k = 2$, replace the first A with the numbers 1 0 and the last A with the number 1. If $k \geq 3$, replace the first and the last A's in the string with the number 1, replace the second A with the numbers 0 1 0, and if there are any more A's, replace each of them with the numbers 1 0. The resulting string $< s_1 s_2 ... s_n >$ is a string of k ones and $(n-k)$ zeros with no consecutive ones. The corresponding subset S of $[n]$ is given by $S = \{i \in [n] : s_i = 1\}$. Since the string has no consecutive ones, S has no consecutive integers. Hence there are $\binom{n+1-k}{k}$ such subsets S. We have derived the following theorem.

Theorem 1. The number c of size-k subsets of $[n]$ that contain no consecutive integers is given by

$$c = c(n, k) = \binom{n+1-k}{k}. \qquad (4)$$

Example. Suppose $n = 7$ and $k = 3$. Then $c(7, 3) = \binom{8-3}{3} = \binom{5}{3} = 10$.
Look at all the shufflings of AAABB:

Shuffling	Resulting Binary Strings	Corresponding Subsets
AAABB	1010100	$\{1, 3, 5\}$
AABAB	1010010	$\{1, 3, 6\}$
AABBA	1010001	$\{1, 3, 7\}$
ABAAB	1001010	$\{1, 4, 6\}$
ABABA	1001001	$\{1, 4, 7\}$
ABBAA	1000101	$\{1, 5, 7\}$
BAAAB	0101010	$\{2, 4, 6\}$
BAABA	0101001	$\{2, 4, 7\}$
BABAA	0100101	$\{2, 5, 7\}$
BBAAA	0010101	$\{3, 5, 7\}$

◇

A Table of Initial Values for $c(n,k)$, $0 \le k \le \lfloor \frac{n+1}{2} \rfloor$.

$n \backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2						
3	1	3	1					
4	1	4	3					
5	1	5	6	1				
6	1	6	10	4				
7	1	7	15	10	1			
8	1	8	21	20	5			
9	1	9	28	35	15	1		
10	1	10	36	56	35	6		
11	1	11	45	84	70	21	1	
12	1	12	55	120	126	56	7	
13	1	13	66	165	210	126	28	1

Note. The diagonals form rows of Pascal's triangle and the recursive formula $c(n,k) = c(n-1,k) + c(n-2,k-1)$ holds. The *Maple* code to generate the values in the table above is given by

```
>seq(seq(binomial(n+1-k,k),k=0..floor(n/2+1/2)),n=0..13);
```

Generating functions for $c(n,k)$

The generating function g_k for $c(n,k)$ when k is fixed is given by

$$g_k(x) = \frac{x^{2k-1}}{(1-x)^{k+1}}. \tag{5}$$

Derivation.

$$\begin{aligned} g_k(x) &= \sum_{n=0}^{\infty} c(n,k)x^n = \sum_{n=2k-1}^{\infty} \binom{n+1-k}{k} x^n = \sum_{j=0}^{\infty} \binom{j+k}{k} x^{j+2k-1} \\ &= x^{2k-1} \sum_{j=0}^{\infty} \binom{j+k}{k} x^j = x^{2k-1} \left(\frac{1}{1-x}\right)^{k+1} \\ &= \frac{x^{2k-1}}{(1-x)^{k+1}} \end{aligned}$$

$\diamond$

These generating functions generate the column sequences for the $c(n,k)$ table of values. For example,

$$g_3(x) = \frac{x^5}{(1-x)^4} = x^5 + 4x^6 + 10x^7 + 20x^8 + 35x^9 + 56x^{10} + 84x^{11} + 120x^{12} + 165x^{13} + ...$$

III. The Number of Subsets of $\{1, ..., n\}$ that Contain No Consecutive Integers

To find the number $c(n)$ of all subsets of $[n]$ that contain no consecutive integers, we sum $c(n,k)$ over all subset sizes k :

$$c(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} c(n,k) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} = F_{n+2}, \quad (6)$$

the $(n+2)$nd Fibonacci number. We note that $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$, and therefore

$$c(n) = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}}. \quad (7)$$

Table of Initial Values for $c(n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$c(n)$	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Note that $c(n)$ satisfies the recursive formula $c(n+2) = c(n+1) + c(n)$ with $c(0) = 1$ and $c(1) = 2$.

Generating function for $c(n)$

$$g(x) = \sum_{n=0}^{\infty} c(n)x^n = x\sum_{n=0}^{\infty} c(n-1)x^{n-1} + x^2\sum_{n=0}^{\infty} c(n-2)x^{n-2}$$

$$= x\,(\sum_{n=1}^{\infty} c(n-1)x^{n-1} + c(-1)x^{-1})$$

$$+ x^2(\sum_{n=2}^{\infty} c(n-2)x^{n-2} + c(-2)x^{-2} + c(-1)x^{-1})$$

$$= x(g(x) + \tfrac{1}{x}) + x^2(g(x) + 0 + \tfrac{1}{x})$$

$$= xg(x) + 1 + x^2 g(x) + x,$$

which implies $g(x) = \frac{1+x}{1-x-x^2}$. (8)

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597

F(n)=((1+sqrt(5))^n-(1-sqrt(5))^n)/(2^n*sqrt(5)).

1, 1, 1, 1, 2, 1, 3, 1, 1, 4, 3, 1, 5, 6, 1, 1, 6, 10, 4, 1, 7, 15,

10, 1, 1, 8, 21, 20, 5, 1, 9, 28, 35, 15, 1, 1, 10, 36, 56,

35, 6, 1, 11, 45, 84, 70, 21, 1, 1, 12, 55, 120, 126, 56, 7,

1, 13, 66, 165, 210, 126, 28, 1