No-Less Functions from $\{1, 2, ..., k\}$ to $\{1, 2, ..., n\}$

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A function f that has $f(x) \geq x$ for all x in the domain of f is called a *no-less function*. We consider no-less functions from $\{1, 2, .., k\}$ to $\{1, 2, ..., n\}$ where $0 \leq k \leq n$. For the case $k = 0$, we count the null function as a no-less function since it vacuously satisfies the definition of a no-less function. We note that no-less functions need not be monotonic functions.

Let $T(n, k) =$ number of functions $f : [k] \rightarrow [n]$ where $f(x) \geq x$ for all x. Note that there are n choices for $f(1)$, $n - 1$ choices for $f(2)$, ..., and $(n - k + 1)$ choices for $f(k)$. Hence we obtain

$$\begin{aligned} T(n, k) &= n(n-1) \cdot \cdot \cdot (n - (k-1)) \\ &= \tfrac{n!}{(n-k)!}. \end{aligned} \quad (1)$$

Example. For $k = 2$ and $n = 4$, $T(4, 2) = 12$ since there are 12 functions $f : [2] \rightarrow [4]$ with $f(1) \geq 1$ and $f(2) \geq 2$, namely,

function	$f(1)$	$f(2)$
f_1	1	2
f_2	1	3
f_3	1	4
f_4	2	2
f_5	2	3
f_6	2	4
f_7	3	2
f_8	3	3
f_9	3	4
f_{10}	4	2
f_{11}	4	3
f_{12}	4	4

Example. Ten runners ran a mile race 10 years ago. Their memories are faulty but not to the extent that they would recall their rankings lower than their actual rankings. How many different accounts of the race rankings could occur?
Answer. $T(10, 10) = 10!$

We provide a table of the initial values of $T(n,k)$ below.

Values of $T(n,k)$

$n\backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	2						
3	1	3	6	6					
4	1	4	12	24	24				
5	1	5	20	60	120	120			
6	1	6	30	120	360	720	720		
7	1	7	42	210	840	2520	5040	5040	
8	1	8	56	336	1680	6720	20160	40320	40320

We note that triangle $T(n,k) = n!(n-k)!$ is given in sequence A008279 of the *Online Encyclopedia of Integer Sequences* at http://oeis.org/A008279. Some other sequences contained in the table above include

Sequence	**Description**
A000012	$a(n) = T(n,0) = 1$, the simplest sequence of positive integers
A000027	$a(n) = T(n,1) = n,$ the natural numbers
A002378	$a(n) = T(n+1,2) = n(n+1),$ oblong numbers
A007531	$a(n) = T(n,3) = n(n-1)(n-2)$
A052762	$a(n) = T(n,4) = n!/(n-4)!,$ products of 4 consecutive integers.
A000142	$a(n) = T(n,n) = n!,$ factorial numbers

Generating functions for $n!/(n-k)!$ with k fixed

• Generating function G_k for $T(n,k)$ with k fixed :

$$G_k(x) = \sum_{n=0}^{\infty} T(n,k)x^n$$

$$= k!\sum_{n=0}^{\infty} \binom{n}{k} x^n$$

$$= \frac{k!x^k}{(1-x)^{k+1}}.$$

• Exponential generating function g_k for $T(n,k)$ with k fixed :

$$g_k(x) = \sum_{n=0}^{\infty} T(n,k) x^n / n!$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n-k)!} x^n$$

$$= x^k \sum_{n=0}^{\infty} \frac{1}{(n-k)!} x^{n-k}$$

$$= x^k e^x$$

A Bijection from the Symmetric Group S_n and the set of No-Less Functions $F_n = \{f : [n] \rightarrow [n] \mid f(x) \geq x$ for all $x \in [n]\}$

Let $b : S_n \rightarrow F_n$ be defined by $b(\sigma) = f$ with $f(x) = \sigma^k(x)$ where k is the least positive integer such that $\sigma^k(x) \geq x$.

Example. Let $\sigma = (5\,1\,4\;8)(3)(6\,2\,7)$. The corresponding no-less function f is given by

x	1	2	3	4	5	6	7	8
$f(x)$	4	7	3	8	8	7	7	8

since $\sigma^1(1) = 4,\ \sigma^1(2) = 7,\ \sigma^1(3) = 3,\ \sigma^1(4) = 8,$
$\sigma^3(5) = 8$ with $\sigma^1(5) < 5$ and $\sigma^2(5) < 5$,
$\sigma^2(6) = 7$ with $\sigma^1(6) < 6$,
$\sigma^3(7) = 7$ with $\sigma^1(7) < 7$ and $\sigma^2(7) < 7$, and
$\sigma^4(8) = 8$ with $\sigma^k(8) < 8$ for $k = 1, 2, 3$.