# A Note on Non-Surjective Functions from [$n$] to [$k$]

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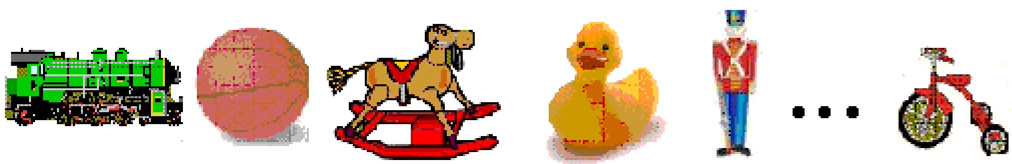
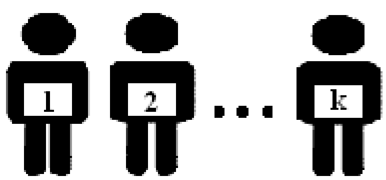
• **An elf**  **has n different toys**



**to distribute among k children**

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**The elf is mean and wants at least one kid to get no toy.**
**In how many ways can he accomplish his goal?**

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Suppose a mean elf has $n$ different toys to distribute among $k$ children. If a child can get more than one toy, in how many ways can the mean elf distribute all the toys and have at least one child not get a toy? We address the answer to this question following the example below.

**Example.** Suppose there are 3 toys ($T_1$, $T_2$, and $T_3$) and 3 children. There are 21 ways the mean elf can distribute the toys and have at least one child not get a toy. The distributions are given below.

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_1, T_2, T_3$ | | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | $T_1, T_2, T_3$ | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | | $T_1, T_2, T_3$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_1, T_2$ | $T_3$ | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_1, T_2$ | | $T_3$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_1, T_3$ | $T_2$ | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_1, T_3$ | | $T_2$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_2, T_3$ | $T_1$ | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_2, T_3$ | | $T_1$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_1$ | $T_2, T_3$ | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_1$ | | $T_2, T_3$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_2$ | $T_1, T_3$ | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_2$ | | $T_1, T_3$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_3$ | $T_1, T_2$ | |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | $T_3$ | | $T_1, T_2$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | $T_1, T_2$ | $T_3$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | $T_1, T_3$ | $T_2$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | $T_2, T_3$ | $T_1$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | $T_1$ | $T_2, T_3$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | $T_2$ | $T_1, T_3$ |

| Child | 1 | 2 | 3 |
|---|---|---|---|
| Toys | | $T_3$ | $T_1, T_2$ |

◇

Let $[n]$ denote the set $\{1, 2, ..., n\}$. Consider a function $f : [n] \to [k]$ that is non-surjective, that is, the range of $f$ is a proper subset of the codomain $[k]$. If $k > n$, then all $k^n$ functions from $[n]$ to $[k]$ are non-surjective.

We now consider the case $2 \leq k \leq n$ and derive the cardinalities of the sets of all such non-surjective functions. This is readily done since the number of surjective functions is well-known ,,, $\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$. Therefore, letting $c(n, k)$, denote the cardinality of the set of all non-surjective functions from $[n]$ to $[k]$, we obtain

$$c(n.k) = k^n - \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

$$= k^n - (k^n + \sum_{j=1}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

$$= \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (k-j)^n.$$

**Triangle of values for $c(n,k)$**

| $n\backslash k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | | | | | | | | |
| 3 | 2 | 21 | | | | | | | |
| 4 | 2 | 45 | 232 | | | | | | |
| 5 | 2 | 93 | 784 | 3005 | | | | | |
| 6 | 2 | 189 | 2536 | 13825 | 45936 | | | | |
| 7 | 2 | 381 | 7984 | 61325 | 264816 | 818503 | | | |
| 8 | 2 | 765 | 24712 | 264625 | 1488096 | 5623681 | 16736896 | | |
| 9 | 2 | 1533 | 75664 | 119005 | 8172576 | 38025127 | 132766208 | 387057609 | |
| 10 | 2 | 3069 | 230056 | 4662625 | 44030736 | 252840049 | 1043501824 | 3470454801 | 9996371200 |

In particular, $c(n,n) = n^n - n!$. See integer sequence A199656 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) at http://oeis.org/A199656.

Appendix

*Maple Code*

```
> N:=(n,k)->sum((-1)^(j-1)*binomial(k,j)*(k-j)^n,j=1..k);
                          k
                        -----
                         \        (j - 1)                     n
         N := (n, k) ->   )   (-1)        binomial(k, j) (k - j)
                         /
                        -----
                        j = 1

> seq(seq(N(n,k),k=2..n),n=2..10);

  2, 2, 21, 2, 45, 232, 2, 93, 784, 3005, 2, 189, 2536, 13825, 45936,
        2, 381, 7984, 61325, 264816, 818503, 2, 765, 24712, 264625,
        1488096, 5623681, 16736896, 2, 1533, 75664, 1119005, 8172576,
        38025127, 132766208, 387057609, 2, 3069, 230056, 4662625,
        44030736, 252840049, 1043501824, 3470454801, 9996371200
```

or

```
> with(combinat, stirling2);
                             [stirling2]

> M:=(n,k)->k^n-k!*stirling2(n,k);

                                n
                M := (n, k) -> k  - k! stirling2(n, k)

> seq(seq(M(n,k),k=2..n),n=2..10);

  2, 2, 21, 2, 45, 232, 2, 93, 784, 3005, 2, 189, 2536, 13825, 45936,
        2, 381, 7984, 61325, 264816, 818503, 2, 765, 24712, 264625,
        1488096, 5623681, 16736896, 2, 1533, 75664, 1119005, 8172576,
        38025127, 132766208, 387057609, 2, 3069, 230056, 4662625,
        44030736, 252840049, 1043501824, 3470454801, 9996371200
```