A Note on Permutations of $[n]$ with Exactly $m$ Cycles and with Elements 1, 2, ..., $m$ in Different Cycles

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Suppose we have $(m - 1)$ letters $A$ and $(n - m)$ letters $B$. It is well known that the number of “words” that can be made using all these letters is $\binom{n-1}{m-1}$, which simplifies to $\binom{n-1}{n-m}$.

Consider one of these words, for example, $B\ B\ A\ B\ B\ A\ A\ B$. Here we have $m = 4$ since there are 3 A's and $n = 9$ since there are 5 B's.

Now starting from the left side of the word replace the first A with the symbols “) ( 2”, replace the second A with “) ( 3”, etc. We now have $B\ B\ (2\ B\ B\ )\ (3\ )\ (4\ B)$.

Next affix to the beginning of the transformed word the symbol “( 1” and append to the end the symbol “)”). We obtain

$$(1\ B\ B\ )\ (2\ B\ B\ )\ (3\ )\ (4\ B)$$.

Our final step is to replace each B with a different number from the set $\{5, 6, 7, 8, 9\}$. There are $5!$ ways to do this. For example, we might obtain

$$(1\ 6\ 8\ )\ (2\ 9\ 5\ )\ (3\ )\ (4\ 7)$$,

which is a permutation of $\{1, 2, ..., 9\}$ written in cyclic form with exactly four cycles and with the elements 1, 2, 3, and 4 in different cycles.

The method above works for any positive integers $m$ and $n$. In effect, we have the following result.

**Theorem.** The number $c(n, m)$ of permutations of $\{1, 2, ..., n\}$ with exactly $m$ cycles and with elements 1, 2, ..., $m$ in different cycles is given by

$$c(n, m) = \binom{n-1}{m-1}(n - m)! = \frac{(n-1)!}{(m-1)!}.$$
Example. Here is a list of the \( c(6, 4) = 20 \) permutations of \( \{0, 1, 2, 3, 4, 5\} \) with exactly 4 cycles and with elements 0, 1, 2, and 3 in different cycles.

\[
\begin{align*}
(0, 5)(1, 4)(2)(3) & \quad (0, 4)(1, 5)(2)(3) & \quad (0, 5)(1, 2, 4)(3) \\
(0)(1, 5)(2, 4)(3) & \quad (0, 4)(1, 2, 5)(3) & \quad (0)(1, 4, 2)(3) \\
(0, 5)(1)(2, 3, 4) & \quad (0)(1, 5)(2)(3, 4) & \quad (0, 4)(1)(2, 3, 5) \\
(0)(1, 4)(2)(3, 5) & \quad (0)(1, 2, 4)(3, 5) & \quad (0)(1, 4)(2)(3) \\
(0, 4, 5)(1)(2, 3) & \quad (0)(1, 5, 4)(2, 3) & \quad (0)(1, 4, 5)(2)(3) \\
(0)(1)(2, 3)(4, 5) & \quad (0)(1)(2, 5, 4)(3) & \quad (0)(1, 2, 4, 5)(3) \\
(0)(1)(2, 3, 4, 5) & \quad (0)(1)(2, 4, 5)(3) & \quad (0)(1)(2)(3, 4, 5) \\
(0)(1)(2)(3, 4, 5) & \quad (0)(1)(2)(3, 4, 5) & \quad (0)(1)(2)(3) \\
\end{align*}
\]

Below is a partial table for the count \( c(n, m) \).

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</table>

* We note the On-Line Encyclopedia of Integer Sequences has the above table of values in sequence \( A094587 \), but indexed with starting values 0 and 0 and giving \( T(n, k) = n!/k! \). Hence we can offer the following interpretation for \( T(n, k) \): the number of permutations on \( n + 1 \) letters with exactly \( k + 1 \) cycles and with the first \( k + 1 \) letters appearing in different cycles.

Example. \( T(3, 1) = 6 \) since there are exactly 6 permutations of \( \{1, 2, 3, 4\} \) with exactly 2 cycles and with elements 1 and 2 in different cycles, namely,

\[
(1)(2 \ 3 \ 4), \quad (1)(2 \ 4 \ 3), \quad (1 \ 3)(2 \ 4), \quad (1 \ 4)(2 \ 3), \quad (1 \ 3 \ 4)(2), \quad (1 \ 4 \ 3)(2).
\]