A Note on Permutations of $[n]$ with Exactly m Cycles and with Elements 1, 2, ..., m in Different Cycles

Dennis Walsh, Middle Tennessee State University

Suppose we have $(m-1)$ letters A and $(n-m)$ letters B. It is well known that the number of "words" that can be made using all these letters is $\binom{m-1+n-m}{m-1}$, which simplifies to $\binom{n-1}{m-1}$.

Consider one of these words, for example,

B B A B B A A B.

Here we have $m = 4$ since there are 3 A's and $n = 9$ since there are 5 B's.
Now starting from the left side of the word replace the first A with the symbols ") (2", replace the second A with ") (3", etc. We now have

B B) (2 B B) (3) (4 B .

Next affix to the beginning of the transformed word the symbol "(1" and append to the end the synbol ")". We obtain

(1 B B) (2 B B) (3) (4 B) .

Our final step is to replace each B with a different number from the set $\{5,\ 6,\ 7,\ 8,\ 9\}$. There are $5!$ ways to do this. For example, we might obtain

(1 6 8) (2 9 5) (3) (4 7),

which is a permutation of $\{1, 2, ..., 9\}$ written in cyclic form with exactly four cycles and with the elements $1, 2, 3,$ and 4 in different cycles.

The method above works for any positive integers m and n. In effect, we have the following result.

Theorem. The number $c(n,m)$ of permutations of $\{1, 2, ..., n\}$ with exactly m cycles and with elements $1, 2, ..., m$ in different cycles is given by

$$c(n,m) = \binom{n-1}{m-1}(n-m)! = \frac{(n-1)!}{(m-1)!}.$$

Example. Here is a list of the $c(6,4) = 20$ permutations of $\{0,1,2,3,4,5\}$ with exactly 4 cycles and with elements $0, 1, 2,$ and 3 in different cycles.

(0, 5)(1, 4)(2)(3)
(0)(1, 5)(2, 4)(3)
(0, 5)(1)(2)(3, 4)
(0)(1, 4)(2)(3, 5)
(0, 4, 5)(1)(2)(3)
(0)(1)(2, 3)(4, 5)
(0)(1)(2)(3, 5, 4)

(0, 4)(1, 5)(2)(3)
(0, 4)(1)(2, 5)(3)
(0)(1, 5)(2)(3, 4)
(0)(1)(2, 4)(3, 5)
(0)(1, 5, 4)(2)(3)
(0)(1)(2, 5, 4)(3)
(0)(1)(2)(3, 4, 5)

(0, 5)(1)(2, 4)(3)
(0)(1, 4)(2, 5)(3)
(0, 4)(1)(2)(3, 5)
(0, 5, 4)(1)(2)(3)
(0)(1, 4, 5)(2)(3)
(0)(1)(2, 4, 5)(3)

Below is a partial table for the count $c(n,m)$.

The count $c(n,m)$ of permutations of $[n]$ with m cycles and with 1, ..., m in different cycles

$n\backslash m$	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	2	2	1					
4	6	6	3	1				
5	24	24	12	4	1			
6	120	120	60	20	5	1		
7	720	720	360	120	30	6	1	
8	5040	5040	2520	840	210	42	7	1

* We note the On-Line Encyclopedia of Integer Sequences has the above table of values in sequence $A094587$, but indexed with starting values 0 and 0 and giving $T(n,k) = n!/k!$. Hence we can offer the following interpretation for $T(n,k)$: the number of permutations on $n+1$ letters with exactly $k+1$ cycles and with the first $k+1$ letters appearing in different cycles.

Example. T(3, 1) $=$ 6 since there are exactly 6 permutations of {1, 2, 3, 4} with exactly 2 cycles and with elements 1 and 2 in different cycles, namely,

(1)(2 3 4), (1)(2 4 3), (1 3)(2 4), (1 4)(2 3), (1 3 4)(2), (1 4 3)(2).