The Correlation for a Power Curve on Nonnegative Support

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Theorem. Let random variable X have a uniform distribution on the interval $[0, c]$ where c is a positive constant. Then, for positive real n, the Pearson product-moment correlation coefficient ρ_n between X and X^n is given by

$$\rho_n = \frac{\sqrt{6n+3}}{n+2},$$

a function of n alone.

Proof. For random variables Y and Z, let $E(Y)$ denote the expected value of Y, let σ_Y denote the standard deviation of Y, let $\text{Var}(Y)$ denote the variance of Y, and let $\text{Cov}(Y, Z)$ denote the covariance of Y and Z. Then

$$\rho_n = \frac{\text{Cov}(X, X^n)}{\sigma_X \cdot \sigma_X n} = \frac{E(X \cdot X^n) - E(X)E(X^n)}{\sqrt{\text{Var}(X)\text{Var}(X^n)}} = \frac{E(X^{n+1}) - E(X)E(X^n)}{\sqrt{[E(X^2) - E(X)^2][E(X^{2n}) - E(X^n)^2]}}.$$

Since $E(X^k) = \frac{1}{c}\int_0^c x^k dx = \frac{1}{c}\left[\frac{1}{k+1}x^{k+1}\right]_0^c = \frac{1}{c}\frac{c^{k+1}}{k+1} = \frac{c^k}{k+1}$,
we obtain

$$\rho_n = \frac{\frac{c^{n+1}}{n+2} - \frac{c}{2}\left(\frac{c^n}{n+1}\right)}{\sqrt{\left(\frac{c^2}{3} - \frac{c^2}{4}\right)\left(\frac{c^{2n}}{2n+1} - \frac{c^{2n}}{(n+1)^2}\right)}}$$

$$= \frac{\frac{1}{n+2} - \frac{1}{2(n+1)}}{\sqrt{\left(\frac{1}{3} - \frac{1}{4}\right)\left(\frac{1}{2n+1} - \frac{1}{(n+1)^2}\right)}}$$

$$= \frac{\frac{n}{2(n+1)(n+2)}}{\sqrt{\frac{1}{12}\left(\frac{n^2}{(2n+1)(n+1)^2}\right)}}$$

$$= \frac{\frac{1}{2(n+2)}}{\sqrt{\frac{1}{12(2n+1)}}}$$

$$= \frac{\sqrt{6n+3}}{n+2} \qquad \square$$

***Corollary*.** The coefficient of determination between X and X^n, ρ_n^2, is given by

$$\rho_n^2 = \frac{6n+3}{(n+2)^2}.$$

In terms of integer sequences from the *On-Line Encyclopedia of Integer Sequences* $(OEIS)$, we have

$$\rho_n^2 = \frac{\text{A016945}(n)}{\text{A000290}(n+2)}.$$