A Short Note on Unsigned Stirling Numbers

Dennis Walsh
Middle Tennessee State University

The unsigned Stirling numbers \(|s(n, k)|\), the absolute values of Stirling numbers of the first kind, are well known to represent the number of permutations on \(n\) elements with exactly \(k\) cycles. For example, \(|s(8, 6)| = 322\) since there are 322 permutations of \(\{1, 2, \ldots, 8\}\) that have exactly 6 cycles. Interestingly, if one takes each size-2 subset of \(\{1, 2, \ldots, 7\}\), multiplies the two elements, and then sums the products, the resulting sum is also 322. The table below illustrates this result.

<table>
<thead>
<tr>
<th>Subset elements</th>
<th>1,2</th>
<th>1,3</th>
<th>1,4</th>
<th>1,5</th>
<th>1,6</th>
<th>1,7</th>
<th>2,3</th>
<th>2,4</th>
<th>2,5</th>
<th>2,6</th>
<th>2,7</th>
</tr>
</thead>
<tbody>
<tr>
<td>product</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
</tr>
<tr>
<td>Subset elements</td>
<td>3,4</td>
<td>3,5</td>
<td>3,6</td>
<td>3,7</td>
<td>4,5</td>
<td>4,6</td>
<td>4,7</td>
<td>5,6</td>
<td>5,7</td>
<td>6,7</td>
<td>Σ-products</td>
</tr>
<tr>
<td>product</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>30</td>
<td>35</td>
<td>42</td>
<td>322</td>
</tr>
</tbody>
</table>

The two routes to the number 322 above suggests a generalization. In fact, for \(n > k \geq 1\), if one takes each size \((n - k)\) subset of \(\{1, 2, \ldots, n - 1\}\), multiplies all the elements, and then sums the products, the resulting sum is equal to the unsigned Stirling number \(|s(n, k)|\). The following theorem formalizes this result.

**Theorem.** For \(1 \leq k < n\), let \(|s(n, k)|\) denote an unsigned Stirling number of the first kind, and let \(A = \{a_1, a_2, \ldots, a_{n-k}\}\) denote a size \((n - k)\) subset of \(\{1, 2, \ldots, n - 1\}\). Then

\[
|s(n, k)| = \sum_{A} (a_1 a_2 \cdots a_{n-k})
\]

where the sum is over all \(\binom{n-1}{n-k}\) subsets \(A\).

**Proof.** Using the well-known fact that the generating function of the unsigned Stirling numbers \(|s(n, k)|\), when \(n\) is fixed, is given by

\[
t(t + 1)(t + 2) \cdots (t + n - 1) = \sum_{k=1}^{n} |s(n, k)| t^k.
\]

Upon expanding \(t(t + 1)(t + 2) \cdots (t + n - 1)\), the coefficient of \(t^k\) is equal to the sum of \(\binom{n-1}{n-k}\) products, each product consisting of \((n - k)\) different factors from \(\{1, \ldots, n - 1\}\) and \(k\) factors of one (the coefficients of the \(t's\)). Hence \(|s(n, k)| = \sum_{A} (a_1 a_2 \cdots a_{n-k})\) where the sum is over all \(\binom{n-1}{n-k}\) size \((n - k)\) subsets \(A\) of \(\{1, 2, \ldots, n - 1\}\). 

\(\Box\)