Notes on Symmetric Subsets of {1, 2, ..., n}

Dennis Walsh
Middle Tennessee State University

A subset A of $\{1, ..., n\}$ is **symmetric** if $k \in A$ implies $(n-k+1) \in A$. For example, $\{1, 3, 4, 6\}$ is a symmetric subset of $\{1, ..., 6\}$.

For even n, there are $2^{n/2}$ symmetric subsets of $\{1, ..., n\}$. To show this, for every subset B of $\{1, ..., n/2\}$, let $C = \{n-b+1 : b \in B\}$. Then every symmetric subset of $\{1, ..., n\}$ is of the form $A = B \cup C$. Since there are $2^{n/2}$ subsets of $\{1, ..., n/2\}$, there are $2^{n/2}$ symmetric subsets of $\{1, ..., n\}$.

For odd n, there are $2^{(n-1)/2}$ symmetric subsets of $\{1, ..., n\}$ that do not contain the middle element $(n+1)/2$. [To show this, apply the same argument as used for even n.] If we add $(n+1)/2$ to the symmetric sets without it, we obtain $2^{(n-1)/2}$ symmetric subsets of $\{1, ..., n\}$ that contain the middle element $(n+1)/2$. Thus there are $2^{(n+1)/2}$ symmetric subsets of $\{1, ..., n\}$ when n is odd.

If a_n denotes the number of symmetric subsets of $\{1, ..., n\}$ for positive integer n and if we let $a_0 = 1$ (since the empty set is vacuously symmetric), then a_n is given by the formula

$$a_n = 2^{\lfloor \frac{n+1}{2} \rfloor}.$$

A recursive formula for a_n is given by $a_0 = 1$, $a_n = a_{n-1}$ for even n and $a_n = 2a_{n-1}$ for odd n.
In *Maple*, to obtain the first 23 entries of the sequence, use the following command:

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>seq(2^floor((n+1)/2),n=0..22);
1, 2, 2, 4, 4, 8, 8, 16, 16, 32, 32, 64, 64, 128, 128, 256, 256, 512, 512, 1024, 1024, 2048, 2048
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Consider a length-n binary string of 0's and 1's. Such a string is symmetric if, whenever entry k is 1, entry $n-k+1$ is also 1. Therefore a_n defined above is also the number of symmetric binary strings of length n. [Note that the empty string is symmetric since it fails to be non-symmetric.].

A generating function g for a_n is given by $g(t) = \sum_{n=0}^{\infty} a_n t^n$ for appropriate t in a neighborhood of zero. We obtain

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} 2^{\lfloor \frac{n+1}{2} \rfloor} t^n \\ &= \sum_{n=0}^{\infty} 2^n t^{2n} + \sum_{n=0}^{\infty} 2^{n+1} t^{2n+1} \\ &= \frac{1}{1-2t^2} + \frac{2t}{1-2t^2} \\ &= \frac{1+2t}{1-2t^2}. \end{aligned}$$