Notes on Acyclic-Function Digraphs

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• A function digraph is a labeled directed graph that satisfies the following two properties:
  (i) the maximum outdegree is 1 for all vertices
  (ii) if $d_v(x) < d_v(y)$, then $x > y$

where $d_v(v)$ denotes the outdegree of vertex $v$.

• An acyclic-function digraph is a function digraph that has no cycles and no loops. In other words, an acyclic function digraph is a labeled digraph that satisfies the following 3 properties:
  (i) the maximum outdegree is 1 for all vertices
  (ii) if $d_v(x) < d_v(y)$, then $x > y$
  (iii) the digraph has no cycles and no loops.

• For an acyclic-function digraph, a vertex with outdegree 0 will be called a root or root vertex. A vertex with outdegree 1 will be called a non-root vertex.

Example.

![Diagram of acyclic-function digraph]

Also, as seen in the example above, a function digraph is a labeled rooted forest in which the label of a root is always greater than any other non-root label.

• For notation, we will let $< x, y >$ denote the graph segment consisting of vertices $x$ and $y$ and the arc from $x$ to $y$, and let $< z >$ denote the isolated (degree 0) root $z$. Then, for example, the set \{ $< 1, 4 >$, $< 2, 4 >$, $< 4, 5 >$, $< 3, 5 >$, $< 6 >$ \} denotes the acyclic-function digraph which has 1 mapped to 4, 2 also mapped to 4, 4 mapped to 5, also 3 mapped to 5, and 6 is an isolated root. Note that we can also
represent this graph as function \( f \) from domain \( D = \{1, 2, 3, 4\} \) to codomain \( C = \{1, 2, 3, 4, 5, 6\} \) where \( f(1) = 4, f(2) = 4, f(3) = 5, \) and \( f(4) = 5 \).

Consider a function \( f \) from domain \( D = \{1, 2, \ldots, n\} \) to codomain \( C = \{1, 2, \ldots, n + m\} \). A length-\( k \) cycle of function \( f \) is a sequence \( < x_1, \ldots, x_k > \) of distinct elements of the domain \( D \) such that \( f(x_i) = x_{i+1} \) and \( f(x_k) = x_1 \). For example, the function \( f \) defined by \( \{(x, f(x)) : (1, 4), (2, 6), (3, 1), (4, 3)\} \) has the 3-cycle \( < 1, 4, 3 > \).

A function \( f \) that has no cycles is called an acyclic function. If \( f \) is an acyclic function, then for every element \( x \) of the domain \( D \) there exists a \( k \) such that \( f^k(x) \in \{n + 1, \ldots, n + m\} \). In other words, an acyclic function “eventually sends” under composition every element of the domain to \( C \setminus D \). For ease of notation, let \( E = C \setminus D = \{n + 1, \ldots, n + m\} \). For any subset \( A \) of \( E \), the tree set of \( A \) under \( f \), denoted \( f^{-\infty}(A) \), will consist of all elements of the domain that are eventually-sent by \( f \) to \( A \).

For example, the function \( f \) defined by \( \{(x, f(x)) : (1, 3), (2, 6), (3, 5), (4, 5)\} \) is an acyclic function. Here \( f \) eventually-sends 1, 3, 4 to 5 since \( f(f(1)) = 5, f(3) = 5, \) and \( f(4) = 5 \). Also, here \( f(2) = 6 \). Thus \( f^{-\infty}(5) = \{1, 3, 4\}, f^{-\infty}(6) = 2, \) and \( f^{-\infty}((5, 6)) = \{1, 2, 3, 4\} \). In contrast, the pre-image of 5 under \( f \) is given by \( f^{-1}(5) = \{3, 4\} \), and the pre-image of \( \{5, 6\} \) is given by \( \{2, 3, 4\} \).

For given domain and codomain sizes \( n \) and \( m \), the number of acyclic functions is given in the following result.

**Theorem 1.** The number \( N(m, n) \) of acyclic functions from domain \( D = \{1, 2, \ldots, n\} \) to codomain \( C = \{1, 2, \ldots, n + m\} \) is given by
\[
N(m, n) = m(m + n)^{n - 1}
\]
for \( m \geq 1 \) and \( n \geq 0 \).

**Proof.** We use induction on \( s \) where \( s = n + m \). For \( s = 1 \), we have the single case where \( m = 1 \) and \( n = 0 \). The only function in this case is the null function (or empty set) which has no cycles and \( N(1, 0) = 1(1 + 0)^{0 - 1} = 1 \).

Now assume \( N(k, r - k) = k(k + r - k)^{r - k - 1} \) for all \( r < s \) where \( 1 \leq k \leq r \) and \( s = n + m \). Let \( E = C \setminus D = \{n + 1, n + 2, \ldots, n + m\} \). Each acyclic function \( f \) from \( D \) to \( C \) induces a partition of \( D \) into \( D_1 = f^{-1}(E) \), the set of elements mapped directly to \( E \), and \( D_2 \), those elements of \( D \) not mapped directly into \( E \). Moreover, since \( f \) is acyclic, \( f \) must “eventually-send” (under successive composition) all elements of \( D_2 \) into \( D_1 \) before their “arrival” in \( E \). Hence the submap of \( f \) with restricted domain \( D_2 \) is itself an acyclic map from \( D_2 \) to \( (D_2 \cup D_1) \).

Now let \( k = |D_1| \), the cardinality of the pre-image of \( E \) under \( f \). For a given \( k \), we count the number of acyclic maps from \( \{1, \ldots, n\} \) into \( \{1, \ldots, n + m\} \). There are \( \binom{n}{k} \) ways to choose the \( k \) elements that comprise \( D_1 \), and then \( m^k \) ways to map the \( k \) elements of \( D_1 \) to the \( m \) elements of \( E \). Since the \( n - k \) elements of \( D_2 \) are acyclicly mapped to \( D = D_1 \cup D_2 \) where \( |D_1| = k \) and \( |D_2| = n - k \), there are, by the induction hypothesis, \( N(k, n - k) = k(k + n - k)^{n - k - 1} \) ways to do this. Thus, by summing over \( k \) and noting that \( k \geq 1 \) (since at least one element of \( D \) is mapped
directly into \( E \), we have \( N(m,n) = \sum_{k=1}^{n} \binom{n}{k} m^{k} k^{n-k-1} = m \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)(k-1)!} m^{k-1} n^{n-k} \).

Letting \( j = k - 1 \), we obtain

\[
N(m, n) = m \sum_{j=0}^{n-1} \binom{n-1}{j} m^{j} n^{n-1-j}
\]

\[= m(m + n)^{n-1}\]

\( \square \)

**Theorem 2.** Let \( m \) be a fixed positive integer and let \( N(m,n) \) denote the number of acyclic functions from domain \( D = \{1,2,\ldots,n\} \) to codomain \( C = \{1,2,\ldots,n + m\} \). Then the exponential generating function \( g_m \) of \( N(m,n) \), where \( g_m(t) = \sum_{n=0}^{\infty} \frac{N(m,n)t^n}{n!} \) for all \( t \) in a neighborhood of zero, is given by

\[
g_m(t) = \exp\left( -mW(-t) \right)
\]

where \( W \) denotes Lambert’s \( W \) function.

Before we prove theorem 2, we present the a lemma and a corollary that will be used in the proof. For any set \( S \), let \( |S| \) denote the cardinality of \( S \).

**Lemma.** For positive integer \( s \geq 1 \) and any real \( m \), the following identity holds:

\[
\sum_{k=1}^{s} (-1)^{k-1} \binom{s}{k} (m + s - k)^{s-1} = (m + s)^{s-1}.
\]

**Proof.** Let \( F \) denote the set of all functions from domain \( D = \{1,2,\ldots,s - 1\} \) to codomain \( C = \{1,2,\ldots,m + s\} \). For \( i \in \{1,2,\ldots,s\} \), let \( G_i \) denote the subset of \( F \) which contains the functions whose range does not include \( m + i \), that is, \( G_i = \{ f \in F \mid m + i \notin f(D) \} \). Then, by the principle of inclusion/exclusion, we have

\[
|\bigcup_{i=1}^{s} G_i| = \sum_{i=1}^{s} |G_i| - \sum_{i<j} |G_i \cap G_j| + \sum_{i<j<k} |G_i \cap G_j \cap G_k| - \ldots + (-1)^{s-1} |\bigcap_{i=1}^{s} G_i|
\]

Now, let \( A = \{m + 1, m + 2,\ldots, m + s\} \) and let \( B \) be any subset of \( A \) with cardinality \( k \). The number of functions in \( F \) whose range does not contain any element of \( B \) is clearly \((m + s - k)^{s-1}\). Hence we have

\[
|\bigcup_{i=1}^{s} G_i| = \binom{s}{1} (m + s - 1)^{s-1} - \binom{s}{2} (m + s - 2)^{s-1} \ldots + (-1)^{s-1} \binom{s}{s} (m)^{s-1}
\]
\[(m + s)^{s-1} = \sum_{k=0}^{s} (-1)^{k-1} \binom{s}{k} (m + s - k)^{s-1} \]

But, since the range of any function in \(F\) has at most \(s - 1\) elements, every function of \(F\) is in at least one subset \(G_i\). Hence \(\bigcup_{i=1}^{s} G_i = F\) and so \(|\bigcup_{i=1}^{s} G_i| = (m + s)^{s-1}\). Thus

\[(m + s)^{s-1} = \sum_{k=1}^{s} (-1)^{k-1} \binom{s}{k} (m + s - k)^{s-1} \quad \square\]

**Corollary.** For positive integer \(s \geq 1\) and any real \(m\), \(\sum_{n=0}^{s} (-1)^{s-n} \binom{s}{n} (m + n)^{s-1} = 0\).

**Proof.** By the lemma, \((m + s)^{s-1} = \sum_{k=1}^{s} (-1)^{k-1} \binom{s}{k} (m + s - k)^{s-1}\) for integer \(s \geq 1\) and any real \(m\). After subtracting \((m + s)^{s-1}\) from both sides we get

\[\sum_{k=0}^{s} (-1)^{k-1} \binom{s}{k} (m + s - k)^{s-1} = 0.\]

Now let \(n = s - k\) and then multiply both sides by \((-1)^{n}\) to get the desired result

\[\sum_{n=0}^{s} (-1)^{s-n} \binom{s}{n} (m + n)^{s-1} = 0. \quad \square\]

**Proof of Theorem 2.** Let \(W\) denote Lambert's \(W\) function. By the definition of Lambert's \(W\) function, if \(w = W(t)\), \(w\) satisfies \(we^w = t\) for \(-e^{-1} \leq t < \infty\). Now let \(\lambda = -W(-t)\) so that \(-\lambda = W(-t)\). But, since \(W(t)e^{W(t)} = t\), we have \(W(-t)e^{W(-t)} = -t\), or equivalently, \(-\lambda e^{-\lambda} = -t\). Thus, if \(\lambda = -W(-t)\), then \(e^{-mW(-t)} = e^{\lambda m}\) where \(\lambda\) satisfies \(\lambda e^{-\lambda} = t\). Therefore we need to show

\[g_m(t) = \sum_{n=0}^{\infty} \frac{m(m+n)^{n-1} e^n}{n!} = e^{\lambda m}\]

where \(\lambda\) satisfies \(\lambda e^{-\lambda} = t\).

First we prove the following claim.

**Claim.**

\[\sum_{n=0}^{\infty} \frac{m(m+n)^{n-1} \lambda^n}{n!} e^{-\lambda(n+m)} = 1. \quad (4)\]
Proof of claim. Expanding \( e^{-\lambda(n+m)} \), rearranging terms, re-indexing, and then switching the order of summations, we obtain

\[
\sum_{n=0}^{\infty} \frac{m(m+n)^{n-1}}{n!} \lambda^n e^{-\lambda(n+m)} = \sum_{n=0}^{\infty} \frac{m(m+n)^{n-1}}{n!} \lambda^n \sum_{k=0}^{\infty} \frac{(-1)^k(m+n)^k}{k!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k(m+n)^{n+k-1}}{n!k!} \lambda^{n+k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{s=n}^{\infty} \frac{(-1)^s-n(m+n)^{s-1}}{n!(s-n)!} \lambda^n (m+n)^{s-1} \quad \text{(upon letting } s = n+k)\]

\[
= \sum_{s=0}^{\infty} \sum_{n=0}^{s} (-1)^{s-n} \binom{s}{n} (m+n)^{s-1} \quad \text{(after switching order of summation)}
\]

But, by the corollary, \( \sum_{n=0}^{s} (-1)^{s-n} \binom{s}{n} (m+n)^{s-1} = 0 \) for real \( m \) and any integer \( s \geq 1 \), the right-side above reduces to \( \frac{\lambda^n m(m-1)}{0!} = 1 \), and the claim, identity (4), is proven.

Now, multiplying both sides of identity (4) by \( e^{\lambda m} \), gives us \( \sum_{n=0}^{\infty} \frac{m(m+n)^{n-1}(\lambda e^{-\lambda})^n}{n!} = e^{\lambda m} \), and, since \( \lambda e^{-\lambda} = t \), we obtain \( \sum_{n=0}^{\infty} \frac{m(m+n)^{n-1}}{n!} = e^{\lambda m} \).

\( \square \)

Theorem 3. For positive integers \( n, r, \) and \( s, \)

\[
(r + s)(r + s + n)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} r(r + k)^{k-1} s(s + n - k)^{n-k-1}.
\]

Proof. By Theorem 1, the LHS is the cardinality of the set of acyclic functions from \( D = \{1, 2, \ldots, n\} \) to \( D \cup R \cup S \) where \( R = \{n+1, n+2, \ldots, n+r\} \) and \( S = \{n+r+1, n+r+2, \ldots, n+r+s\} \). We can construct any acyclic function \( f : D \to D \cup R \cup S \) by performing the following steps:

(i) For \( k \in \{0, 1, \ldots, n\} \), choose \( k \) of the \( n \) elements of \( D \) that will be the tree-set of \( R \). The remaining \( n - k \) elements of \( D \) will be the tree-set of \( S \).
(ii) Construct an acyclic map $f_1$ from the tree-set of $R$ to $R$.
(iii) Construct an acyclic map $f_2$ from the tree-set of $S$ to $S$.
(iv) Let $f = f_1 \cup f_2$.

The number of ways to perform the steps is $\binom{n}{k} r(r + k)^{k-1}$, $s(s + n + k - 1)^{n+k-1}$, and $1$, respectively. Hence, by summing $\binom{n}{k} r(r + k)^{k-1} s(s + n + k - 1)^{n+k-1}$ over $k = 0, \ldots, n$, we obtain the number of all acyclic functions from $D$ to $D \cup R \cup S$. □

**Example.** With $n = 5$, $r = 3$, and $s = 2$, consider $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, \ldots, 6, 7, 8, 9, 10\}$ defined by $< f(1), \ldots, f(10) > = < 4, 9, 1, 7, 2 >$.

Let $f_1 : \{1, 3, 4\} \rightarrow \{1, 3, 4, 6, 7, 8\}$ be defined by $< f(1), f(3), f(4) > = < 4, 1, 7 >$ and $f_2 : \{2, 5\} \rightarrow \{2, 5, 9, 10\}$ be defined by $< f(2), f(5) > = < 9, 2 >$. Then $f = f_1 \cup f_2$. 