Evaluating at $t = a$, we see that $m(a)$ divides $k_1^n f\bigl(r(a)\bigr)$. Any common prime divisor of $m(a)$ and k_1^n divides $k_2 b$, but k_1 is relatively prime to b by (2) and also to k_2. Thus $m(a)$ is relatively prime to k_1, so it divides $f\bigl(r(a)\bigr)$, and we are done. ■

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A Curious Way to Test for Primes

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Upon hearing that you are a student of mathematics, a cab driver says to you, "Check this out. The second derivative of e^x is e^x, right? And e^x evaluated at 0 is equal to 1, right? Therefore 2 has got to be a prime number." Your first reaction is a condescending chuckle for the cabby who seems to dabble in mathematics and appears to make a jumble of it. "Well, at least she's logically correct—2 is a prime number," you mumble to yourself with some smugness. But the cab driver hears you and takes a long route to your destination. You end up paying a hefty fare. The cab driver smirks as she drives off.

In fact, the smirking cabby stated a specific case of the theorem we provide below, a theorem which offers an unusual characterization of prime numbers based on differentiation. The cab driver could give another specific case by stating, "The third derivative of $e^x + e^{x^2/2}$ evaluated at $x = 0$ is 1, and thus 3 is a prime number" Offering another example, she could state accurately that, "The fourth derivative of $e^x + e^{x^2/2} + e^{x^3/3}$ evaluated at 0 does not equal 1, and hence 4 is not a prime number." You can probably guess the pattern. We give it below in a theorem with a surprisingly simple proof that uses the series expansion of $e^{x^k/k}$ and the power rule for differentiation.

THEOREM. *For each positive integer $n > 1$, define the function g_n by $g_n(x) = \sum_{k=1}^{n-1} e^{x^k/k}$. A positive integer n is prime if and only if $\frac{d^n}{dx^n} g_n(0) = 1$.*

Proof. Let n be a positive integer greater than 1. Note that $e^{x^k/k}$ has a series expansion given by

$$e^{x^k/k} = \sum_{j=0}^{\infty} \frac{(x^k/k)^j}{j!} = \sum_{j=0}^{\infty} \frac{x^{kj}}{k^j\, j!}$$

for all real x. Hence, we have

$$g_n(x) = \sum_{k=1}^{n-1} \sum_{j=0}^{\infty} \frac{x^{kj}}{k^j\, j!},$$

and upon differentiating g_n we obtain

$$\begin{aligned}\frac{d^n}{dx^n}g_n(x) &= \frac{d^n}{dx^n}\sum_{k=1}^{n-1}\sum_{j=0}^{\infty}\frac{x^{kj}}{k^j j!}\\ &= \sum_{k=1}^{n-1}\sum_{j=0}^{\infty}\frac{1}{k^j j!}\frac{d^n}{dx^n}(x^{kj})\\ &= \sum_{k=1}^{n-1}\sum_{j=0}^{\infty}\frac{(kj)(kj-1)(kj-2)\cdots(kj-n+1)x^{kj-n}}{k^j j!}I[kj \geq n],\end{aligned}$$

where the indicator function $I[s]$ takes the value 1 if statement s is true and the value 0 otherwise. Upon evaluating $\frac{d^n}{dx^n}g_n(x)$ at $x = 0$, we see that every term in the inner sum above vanishes except for the terms where $kj = n$. Thus, using $k|n$ to denote the statement "k divides n," we get

$$\begin{aligned}\frac{d^n}{dx^n}g_n(0) &= \sum_{k=1}^{n-1}\frac{n(n-1)(n-2)\cdots(n-n+1)}{k^{n/k}(n/k)!}I[k|n]\\ &= \sum_{k=1}^{n-1}\frac{n!}{k^{n/k}(n/k)!}I[k|n].\end{aligned}$$

If n is prime, the only divisor of n that is less than or equal to $n-1$ is 1, in which case the summation above collapses to the single term for $k = 1$. Hence, when n is prime, we obtain

$$\frac{d^n}{dx^n}g_n(0) = \frac{n!}{1^{n/1}(n/1)!} = 1.$$

If n is not prime and $n > 1$, there exist positive integers k and r, both in the interval $[2, n-1]$, such that $n = kr$. Thus, if n is not prime and $n > 1$, we have

$$\frac{d^n}{dx^n}g_n(0) \geq 1 + \frac{(kr)!}{k^r r!} > 1.$$

We conclude with *Maple* code below that performs the primality test for small n.

```
eval(diff(sum(exp(t^k/k), k = 1..n − 1), t$n), t = 0);
```

Replace n in the code with the specific positive integer you wish to test. If the output is 1, then n is a prime. If the output is not 1, then is n not a prime.