Problem 1. Consider the binary operation $\sqcup$ defined on the set $\mathbb{Z}$ of integers by the formula

$$m \sqcup n = |m - n|$$

Part (a). Is this operation commutative?

Part (b). Is this operation associative?

Problem 2. Consider the binary operation $\uplus$ defined on the set $\mathbb{R}$ of real numbers by the formula

$$a \uplus b = \sqrt[4]{a^4 + b^4}$$

Part (a). Is this operation commutative?

Part (b). Is this operation associative?

Problem 3. Consider the set $\mathbb{Z}_{10} = \{0,1,2,3,4,5,6,7,8,9\}$ along with the function $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ defined by $f(x) = 4 \boxtimes_{10} x$.

Part (a). Write the function f in tabular notation.

$$f : \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & & & & & & & & & \end{pmatrix}$$

Part (b). Is this function a member of the family $\mathcal{P}_{\mathbb{Z}_{10}}$?

Problem 4. Consider the following members of the permutation group $\boldsymbol{\mathcal{P}}_7$.

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 5 & 1 & 7 & 6 \end{pmatrix} \qquad g : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 2 & 6 & 1 & 7 \end{pmatrix}$$

Part (a). Construct the formulas for the permutations g^{-1}, $f \circ g$, and $g \circ f \circ g^{-1}$.

Part (b). What is the smallest positive integer n such that f^n is the identity for $\boldsymbol{\mathcal{P}}_7$?

Problem 5. Consider the group $\boldsymbol{\mathcal{Z}}_{20} = (\mathbb{Z}_{20}, \boxplus_{20})$.

Part (a). What is the smallest positive integer n such that 8^n is the identity for $\boldsymbol{\mathcal{Z}}_{20}$?

Part (b). What is the largest negative integer m such that 3^m is the identity for $\boldsymbol{\mathcal{Z}}_{20}$?

Problem 6. A binary operation on the set $X = \{a, b, c, d, e\}$ has been partially defined by the table below. If we know that $(X,*)$ is a Sudoku algebra, complete the table.

$*$	a	b	c	d	e
a	a	c	b	e	
b	b	a	e	d	c
c		d	a		e
d	e	b	d	c	a
e	d			a	b

Problem 7. Consider the set $M = \{I, A, B, C, D, E\}$, where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

Recall from linear algebra that the product of two 2×2 matrices is given by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

Part (a). Fill in the table below.

$*$	I	A	B	C	D	E
I						
A						
B						
C						
D						
E						

Part (b). Is $(M,*)$ a Sudoku algebra? Explain.

Part (c). Is $(M,*)$ a group? Explain.

Problem 8. Consider the set $I = \{x \in \mathbb{R} : 0 < x \leq 1\}$ along with the relation

$$a \because b = \begin{cases} ab & \text{if} \quad a < b \\ b/a & \text{if} \quad b \leq a \end{cases}$$

Part (a). Show that this relation is a binary operation on the set I.

Part (b). Show that $x = 2/3$ and $x = 1/6$ are *both* solutions to the equation $(1/2) \because x = 1/3$.

Part (c). Show that $\varepsilon = 1$ serves as an identity for the algebra $(I, \because)$.

Part (d). Show that every element of I serves as its own inverse with respect to the operation $\because$.

Part (e). Without using a counterexample, explain why the operation $\because$ *cannot* be associative. Can you find a specific counterexample?

Problem 9. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group, and suppose we know that there exist $a, b, c \in X$ such that

- $a * c = c * a$,
- $b * c = c * b$, and
- $c = (a * b) * (b * a)^{-1}$.

Use this information to show that $b * (c * a) = a * b$.

Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group with identity ε, and suppose that $a \in X$. We say that a has *finite order* in this group provided there exists a smallest positive integer p such that $a^p = \varepsilon$.

Problem 10. Fill in the gaps in the proof of the following result.

If $\boldsymbol{\mathcal{X}} = (X,*)$ is a finite group, then every member of X has finite order in the group $\boldsymbol{\mathcal{X}}$.

Proof. Let $X = \{\varepsilon, a_1, \dots, a_n\}$, where ε is the identity for $\boldsymbol{\mathcal{X}}$. Let $b \in X$ and consider the set

$$S_b = \{b^j : j \in \mathbb{Z}^+\}$$

There must exist positive integers $j < k$ such that $b^j = b^k$. [WHY?]

Consequently, we know that there exists a positive integer m such that $b^m = \varepsilon$. [WHY?]

It follows that there is a smallest positive integer p such that $b^p = \varepsilon$. [WHY?]

We must therefore conclude that b has finite order in the group $\boldsymbol{\mathcal{X}}$.

Problem 11. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is a group with identity ε, and suppose that $a \in X$ has finite order p. Prove that the elements $a^1, a^2, \dots, a^{p-1}, a^p$ are all distinct. (Hint: Suppose, for purposes of contradiction, that two of these elements are equal.)