A *binary operation* on a set X is a rule that takes any two members of the set X as input and returns another member of the set X as output. The rule may be defined by a table, a formula, or even a graph.

Problem 1. Which of the following formulas defines a binary operation on the set $\mathbb{Q}$ of rational numbers? Explain your thinking on each formula.

$$a \ltimes b = ab \qquad a \asymp b = \sin(a)\cos(b) \qquad a \ast b = \sqrt[3]{a^3 b^4}$$

It is common to use tables to define binary rules on small sets, and we have certain conventions that we use when interpreting such tables. For example, suppose we let $X = \{1,2,3,4\}$ and let $\propto$ be the binary rule defined by the following table.

$\propto$	**1**	**2**	**3**	**4**
1	2	1	4	2
2	3	4	4	1
3	4	1	3	2
4	4	4	2	2

This table is read from left-to-right. In other words, when we consider the term $2 \propto 3$, it is understood that we are looking at the intersection of the "2-row" and the "3-column". Therefore, based on this table,

$$2 \propto 3 = 4$$

Problem 2. Is the rule defined by the table above a binary operation on the set $X = \{1,2,3,4\}$? Explain your thinking.

Problem 3. Let $X = \{0,1,2,3,4,5\}$. Define a binary rule on the set X by

$$a \boxplus_6 b = r$$

where r is the remainder obtained when $a + b$ is divided by 6.

Part (a). What is the value of $5 \boxplus_6 4$?

Part (b). What is the value of $4 \boxplus_6 4$?

Part (c). Fill in the table below using the definition of this binary rule.

$\boxplus_6$	**0**	**1**	**2**	**3**	**4**	**5**
0						
1						
2						
3						
4						
5						

Part (d). Does the rule define a binary operation on the set X?

Commutative Binary Operation on a Set

Let X be a nonempty set, and suppose that $\star$ represents a binary operation on the set X. We say that two elements a and b from the set X *commute* provided $a \star b = b \star a$.

We say that the binary operation $\star$ is *commutative* provided $a \star b = b \star a$ for *every* pair of elements $a, b \in X$.

Problem 4. Let $\mathbb{R}^+$ represent the set of all positive real numbers, and consider the binary rules $\star$ and $\wedge$ defined by the formula

$$a \star b = e^{a/b} \qquad \text{and} \qquad a \wedge b = e^{ab}$$

Part (a). Explain why each rule is an operation on the set $\mathbb{R}^+$.

Part (b). Is either operation commutative? Justify your answer.

Part (c). Does it make sense to say that the set $\mathbb{R}^+$ is commutative? Explain.

Problem 5. Let $X = \{a, b, c, d\}$ and consider the binary operations $\star$ and $\curlywedge$ defined by the tables below. Are either of these operations commutative? Explain your thinking.

$\star$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
$\boldsymbol{a}$	c	d	a	b
$\boldsymbol{b}$	d	c	b	a
$\boldsymbol{c}$	a	b	c	d
$\boldsymbol{d}$	b	a	d	c

$\curlywedge$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
$\boldsymbol{a}$	b	a	d	b
$\boldsymbol{b}$	a	d	d	a
$\boldsymbol{c}$	d	a	c	b
$\boldsymbol{d}$	d	d	b	b

Associative Binary Operation on a Set

Let X be a nonempty set, and suppose that $\star$ represents a binary operation on the set X. We say that the binary operation $\star$ is *associative* provided

$$a \star (b \star c) = (a \star b) \star c$$

for *all* elements $a, b, c \in X$.

Determining whether or not a given binary operation is associative can be challenging; consequently, we often focus on operations that are built from operations that we know are associative.

Associative Operations on the Set of Real Numbers

We will assume that real number addition and multiplication are associative binary operations on the set $\mathbb{R}$ of real numbers.

Problem 6. Let $\mathbb{Z}$ represent the set of integers. Construct a proof to show that the binary operation on the set $\mathbb{Z}$ defined by

$$a \diamond b = a + b - ab$$

is associative. Is this operation commutative?

Let X be any nonempty set. We will let $[X \to X]$ represent the set of all functions $f: X \to X$. Now, if $f, g \in [X \to X]$, we can talk about the *composition* of these functions. Remember, the functions $f \circ g$ and $g \circ f$ are defined by the formulas

$$[f \circ g\,](x) = f\big(g(x)\big) \qquad\qquad [g \circ f\,](x) = g\big(f(x)\big)$$

Function composition serves as a binary operation on the set $[X \to X]$.

Problem 7. Consider the set $[\mathbb{Z} \to \mathbb{Z}]$. The functions $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(x) = 2x^3 - x \qquad\qquad g(x) = x^2 - 4$$

are members of this set. Is it true that $f \circ g = \ g \circ f$? Justify your answer.

When we want to define functions whose domain is a finite set, we commonly use *tabular notation* as a shorthand way to do so. For example, suppose we let $X = \{1,2,3,4\}$. Consider the function $h: X \to X$ defined by

$$h(1) = 1 \quad h(2) = 1 \quad h(3) = 4 \quad h(4) = 4$$

It is common to write this function assignment as

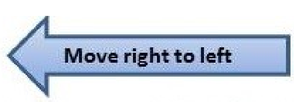
$$h\!: \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}$$

In this notation, the top row represents the domain of the function, and the bottom row represents the image of the domain.

We can compute the composition of two functions from $[X \to X]$ using tabular notation as well. We only need to remember that function composition is read *from right to left*. Suppose $f, g \in [X \to X]$ are defined by

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \quad g : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

The composition $g \circ f$ would be



$$g \circ f : \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

In words, we know that $f(1) = 3$, and we know that $g(3) = 2$. Therefore, we know $g\big(f(1)\big) = 2$. Each column of the tabular notation for $g \circ f$ is filled in this way.

Problem 8. Let $X = \{1,2,3,4\}$ and consider the functions f, g, and h written in tabular notation above.

Part (a). Use tabular notation to construct the composite functions $f \circ g$ and $h \circ f$.

Part (b). Is function composition a commutative operation on the set $[X \to X]$? Explain.

> **THEOREM 1.1**
> Let X be any nonempty set, and consider the family of functions $[X \to X]$. If $f, g, h \in [X \to X]$, then it is always true that $f \circ (g \circ h) = (f \circ g) \circ h$. In other words, the binary operation of function composition on the set $[X \to X]$ is always associative.

Proof. We need to show that, for all $x \in X$, we have $[f \circ (g \circ h)](x) = [(f \circ g) \circ h](x)$. To this end, observe that

$$[f \circ (g \circ h)](x) = f([g \circ h](x)) = f(g(h(x))$$

$$[(f \circ g) \circ h](x) = [(f \circ g)]\big(h(x)\big) = f(g(h(x))$$

QED

Homework.

Problem 1. Consider the binary operation $\|$ on the set $\mathbb{Z}$ of integers defined by $a \| b = |a + b|$.

Part (a). Prove that this operation is commutative.

Part (b). Give a specific counterexample to show that this operation is *not* associative.

Problem 2. Consider the binary operation $\Bumpeq$ on the set $\mathbb{R}$ of real numbers defined by

$$x \Bumpeq y = (x^3 + y^3)^{1/3}$$

Prove that this operation is associative.

Problem 3. Consider the rule defined by $x \between y = (x^2 + y^2)^{1/2}$. Is this rule a binary operation on the set $\mathbb{Z}^+$? Justify your answer.

Problem 4. Let $X = \{0,1,2,3\}$ and for all $a, b \in X$, let $a \boxplus_4 b$ denote the remainder obtained when $a + b$ is divided by 4. Fill in the operation table below.

$\boxplus_4$	**0**	**1**	**2**	**3**
0				
1				
2				
3				

Problem 5. Consider the set $X = \{1,2,3\}$ and the functions defined by

$$\varepsilon : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \alpha : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad \beta : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\gamma : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad \delta : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \zeta : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Fill in the operation table below. (The symbol $\circ$ denotes function composition.)

$\circ$	$\boldsymbol{\varepsilon}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\zeta}$
$\boldsymbol{\varepsilon}$						
$\boldsymbol{\alpha}$						
$\boldsymbol{\beta}$						
$\boldsymbol{\gamma}$						
$\boldsymbol{\delta}$						
$\boldsymbol{\zeta}$						