In this investigation, we will start exploring another way to create new groups from ones we already know.

Set-Operation in Groups

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let A and B be nonempty subsets of X. The *left operation* of the set A on the set B is defined to be the set

$$AB = \{x * y : x \in A, y \in B\}$$

We can also describe this set as the *right operation* of the set B on the set A.

Problem 1. Consider the group $\boldsymbol{\mathcal{S}}_5 = (S_5,\circ)$ of permutations on the five-element set $X = \{1,2,3,4,5\}$ under function composition. Let

$$A = \left\{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 1 & 4 & 2 & 5\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 5 & 3 & 2 & 4 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 3 & 1 & 5 & 2\end{pmatrix}\right\}$$

$$B = \left\{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 1 & 2 & 4 & 5 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 5 & 4 & 1 & 3\end{pmatrix}\right\}$$

Construct the set AB and the set BA.

Problem 2. Consider the group $\boldsymbol{\mathcal{S}}_\times = (S_\times,*)$ of symmetries on the cross. Let $A = \{F, RRRR\}$. For each $x \in S_\times$, construct the sets

$$\{x\}A \qquad \text{and} \qquad A\{x\}$$

What are some things you notice about these sets?

If $\boldsymbol{\mathcal{X}} = (X,*)$ is any group and $A \subseteq X$ is nonempty, then it is customary to let xA and Ax represent the sets $\{x\}A$ and $A\{x\}$, respectively for any $x \in X$.

> ***Theorem 10.1***
>
> Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and suppose that H is a subgroup of $\boldsymbol{\mathcal{X}}$. If we let
>
> $$LC_H = \{aH : a \in X\}$$
>
> then the following statements will always be true:
>
> 1. Every element of X will appear in exactly one member of LC_H.
> 2. There is a bijection between aH and bH for all $a, b \in H$.

Proof of Theorem 10.1

Let ε be the identity for the group $\boldsymbol{\mathcal{X}}$. Since H is a subgroup of $\boldsymbol{\mathcal{X}}$, we know that ε is a member of H. Now, since

$$aH = \{a * x : x \in H\}$$

we know that $a = a * \varepsilon$ must be a member of aH. Consequently, every element of X must appear in *at least one* member of LC_H. We must prove that every element of X appears in *exactly one* member of LC_H.

Problem 3. Fill in the gaps for the following argument.

Let $y \in X$, and suppose that $y \in aH$ and $y \in bH$. We will prove that $aH \subseteq bH$. We know there exist $u, v \in H$ such that $y = a * u$ and $y = b * v$.

[Why do we know this?]

It follows that $a * u = b * v$. Hence, we know that $= b * (v * u^{-1})$, and we must conclude that $a \in bH$.
[Why can we conclude this?]

Now, suppose that $w \in aH$. There must exist some $x \in H$ such that $w = a * x$. Therefore, we also know that $w = b * z$ for some $z \in H$.

[Why can we draw this conclusion?]

We may therefore conclude that $aH \subseteq bH$.

Problem 4. Let $y \in X$, and suppose that $y \in aH$ and $y \in bH$. Prove that $bH \subseteq aH$.

Putting Problems 3 and 4 together, we have shown that, if $y \in aH$ and $y \in bH$, then it must be true that $bH = aH$. Consequently, every element of X must appear in exactly one member of LC_H. This proves Claim (1) of Theorem 10.1.

Problem 5. Let $a, b \in X$ and consider the function $f: aH \to bH$ defined by $f(a * x) = b * x$, where $x \in H$. Prove that the function f is one-to-one and onto. (This establishes Claim (2) of Theorem 10.1.)

Equinumerous Partition of a Set

Let X be any nonempty set. A *partition* of the set X is any family $\mathfrak{F}$ of subsets of X having the property that every element of X will appear in exactly one member of $\mathfrak{F}$. The members of a partition are *pairwise disjoint*. In other words, if A, B are members of a partition $\mathfrak{F}$ and $A \neq B$, then $A \cap B = \emptyset$.

We say that a partition $\mathfrak{F}$ of a set X is *equinumerous* provided there is a bijection between any two members of $\mathfrak{F}$.

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group. If H is a subgroup of $\boldsymbol{\mathcal{X}}$, then Theorem 10.1 tells us the family LC_H forms an equinumerous partition of the universe X. We refer to the members of LC_H as the *left cosets* for the subgroup H in the group $\boldsymbol{\mathcal{X}}$.

It is important to note that we could also consider the family

$$RC_H = \{Ha : a \in X\}$$

It turns out that this family also forms an equinumerous partition of the universe X. (The proof of this statement is virtually identical to the proof of Theorem 10.1.) We refer to the members of RC_H as the *right cosets* for the subgroup H in the group $\boldsymbol{\mathcal{X}}$.

Problem 6. Consider the group $\boldsymbol{\mathcal{S}}_\times = (S_\times,*)$ of symmetries on the cross. Let $H = \{FR, RRRR\}$. Construct the families RC_H and LC_H. Do you notice any shortcuts that you can use?

Problem 7. Consider the group $\mathcal{Z} = (\mathbb{Z}, +)$ and the subgroup $4\mathbb{Z}$.

Part (a). Construct the family of left cosets for the subgroup $4\mathbb{Z}$ in the group $\mathcal{Z}$.

Part (b). Explain why we must have $m(4\mathbb{Z}) = (4\mathbb{Z})m$ for every integer m.

If X is any nonempty *finite* set and $\mathfrak{F}$ is any equinumerous partition of X, then every member of the family $\mathfrak{F}$ must contain exactly the same number of elements (since there is a bijection between any two members of $\mathfrak{F}$). This fact is key to proving a very useful result about finite groups.

> ***Theorem 10.2 (Langrange's Theorem)***
>
> Let $\mathcal{X} = (X,*)$ be any finite group, and suppose that H is a subgroup of $\mathcal{X}$. The number of elements in H must be a divisor of the number of elements in X.

Proof of Theorem 10.2

Suppose that X contains exactly n elements, and form the equinumerous partition LC_H. Since H is closed under the operation $*$, we know that $H \in LC_H$.

[Why do we know this?]

Since X is finite, we know that LC_H contains only finitely many members. Suppose that LC_H contains exactly r members. Each member of LC_H must contain exactly the same number of elements. Let m be this number. It follows that $n = rm$.

[Why can we conclude this?]

We may conclude that m is a divisor of n. Since every member of LC_H contains exactly m elements, and since H is a member of LC_H, we have shown that the number of elements in H must be a divisor of the number of elements in X.

QED

Problem 8. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is any finite group containing at least two elements, and suppose that $a \in X$ is not the identity. If the number of elements in X is *prime*, use Problem 6 from Investigation 9 and Lagrange's Theorem to prove that is $\boldsymbol{\mathcal{X}}$ cyclic with a as a generator.

Problem 9. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let H be a subgroup of $\boldsymbol{\mathcal{X}}$.

Part (a). For each $a \in X$, prove that there is a bijection between the left coset aH and the right coset Ha.

Part (b). If X is *finite*, explain why the families LC_H and RC_H must have the same number of members.

Index of a Subgroup in a Finite Group

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any finite group, and suppose that H is a subgroup of $\boldsymbol{\mathcal{X}}$. The number of members in LC_H (and in RC_H) is called the *index* of the subgroup H in the group $\boldsymbol{\mathcal{X}}$. The index of H in the group $\boldsymbol{\mathcal{X}}$ is traditionally denoted by the symbol

$$[X|H]$$

Homework.

Problem 1. Consider the product group $\mathcal{Z}_6 \times \mathcal{Z}_2 \times \mathcal{Z}_2$ and the subgroup

$$H = \{(0,0,0),(3,0,0),(3,1,1),(0,1,1)\}$$

Part (a). How many distinct left cosets for H are there?

Part (b). Construct the left cosets for H.

Problem 2. Consider the group $\boldsymbol{S}_4 = (S_4, \circ)$ of permutations on the four-element set $X = \{1,2,3,4\}$ under function composition along with the subgroup

$$H = \left\{\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 1 & 2\end{pmatrix}\right\}$$

Part (a). How many distinct left cosets for H are there?

Part (b). Construct the left cosets for H.

Part (c). Construct the right cosets for H.

Problem 3. Let $\boldsymbol{\mathcal{X}} = (X, *)$ be any group, let H be a subgroup of $\boldsymbol{\mathcal{X}}$, and let $x, y \in X$. Prove that $y \in xH$ if and only if $y^{-1} * x \in H$.

The *quaternion group* is an important finite group of 2×2 matrices with complex entries. Let $i = \sqrt{-1}$ and let

$$\varepsilon = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad I = \begin{bmatrix}0 & i\\ i & 0\end{bmatrix}, \quad J = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \quad K = \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}$$

$$-\varepsilon = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, \quad -I = \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}, \quad -J = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \quad -K = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$$

If we let $\text{Qua} = \{\varepsilon, I, J, K, -\varepsilon, -I, -J, -K\}$, then Qua forms a group under (complex) matrix multiplication. The operation table for the quaternion group $\boldsymbol{\mathcal{Qua}} = (\text{Qua}, *)$ is shown below.

$*$	$\boldsymbol{\varepsilon}$	$\boldsymbol{-\varepsilon}$	$\boldsymbol{I}$	$\boldsymbol{-I}$	$\boldsymbol{J}$	$\boldsymbol{-J}$	$\boldsymbol{K}$	$\boldsymbol{-K}$
$\boldsymbol{\varepsilon}$	ε	$-\varepsilon$	I	$-I$	J	$-J$	K	$-K$
$\boldsymbol{-\varepsilon}$	$-\varepsilon$	ε	$-I$	I	$-J$	J	$-K$	K
$\boldsymbol{I}$	I	$-I$	$-\varepsilon$	ε	K	$-K$	$-J$	J
$\boldsymbol{-I}$	$-I$	I	ε	$-\varepsilon$	$-K$	K	J	$-J$
$\boldsymbol{J}$	J	$-J$	$-K$	K	$-\varepsilon$	ε	I	$-I$
$\boldsymbol{-J}$	$-J$	J	F	R	ε	$-\varepsilon$	$-I$	I
$\boldsymbol{K}$	K	$-K$	J	$-J$	$-I$	I	$-\varepsilon$	ε
$\boldsymbol{-K}$	$-K$	K	$-J$	J	I	$-I$	ε	$-\varepsilon$

Problem 4. Is the Quaternion Group isomorphic to the group $\boldsymbol{S}_{\times} = (S_{\times},*)$ of symmetries on the cross? Justify your answer.

Problem 5. Consider the subgroup $H = \{\varepsilon, -\varepsilon\}$.

Part (a). Construct the set of left cosets for H.

Part (b). Construct the set of right cosets for H.

Problem 6. If X is any subgroup of the Quaternion Group, what are the possibilities for the number of elements X could contain?

Problem 7. Identify all of the subgroups of the Quaternion Group.

Problem 8. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is any group, and suppose that H is any subgroup of $\boldsymbol{\mathcal{X}}$. For all $a \in X$ prove that $aH = H$ if and only if $a \in H$.