

Let X be any nonempty set, and suppose that $\theta \subseteq X \times X$. Recall that θ is an *equivalence relation* on the set X provided the following conditions are met:

- The set θ is reflexive; that is, $(a, a) \in \theta$ for all $a \in X$.
- The set θ is symmetric; that is, if $(a, b) \in \theta$ then $(b, a) \in \theta$.
- The set θ is transitive; that is, if $(a, b), (b, c) \in \theta$ then $(a, c) \in \theta$.

Let X be any nonempty set, and suppose that $\theta \subseteq X \times X$ is an equivalence relation on X . For each $a \in X$, let

$$[a]_{\theta} = \{b \in X : (a, b) \in \theta\}$$

These sets are known as the *equivalence classes modulo θ* .

Problem 1. Let X be any nonempty set, and suppose that $\theta \subseteq X \times X$ is an equivalence relation on X . Suppose $a, b \in X$, and suppose that there exist $x \in [a]_{\theta} \cap [b]_{\theta}$.

Part (a). Explain why we know that $(x, a) \in \theta$ and $(b, x) \in \theta$.

Part (b). Use Part (a) to explain why we must conclude that $(b, a) \in \theta$.

Part (c). Now, suppose that $y \in [a]_{\theta}$. Use Part (b) to explain why $(b, y) \in \theta$.

Part (d). Explain why we may conclude that $[a]_{\theta} \subseteq [b]_{\theta}$.

Part (e). Adjust your arguments in Parts (c) and (d) to prove that we must also have $[b]_{\theta} \subseteq [a]_{\theta}$.

In Problem 1, you proved the following result:

If $\theta \subseteq X \times X$ is an equivalence relation on X and there exist $x \in [a]_{\theta} \cap [b]_{\theta}$ then $[a]_{\theta} = [b]_{\theta}$.

This fact, along with the fact that θ is reflexive, tells us the equivalence classes modulo θ form a *partition* of the set X --- that is, every member of X must appear in *exactly one* equivalence class modulo θ .

Let n be a fixed positive integer. In Investigation 2, you showed that two integers a and b have the same remainder when divided by n if and only if $b - a \in n\mathbb{Z}$. In this and subsequent investigations, we will explore the implications of this fact for groups in general.

Problem 2. Let n be a fixed positive integer and define $\theta_n \subseteq \mathbb{Z} \times \mathbb{Z}$ by the rule $(a, b) \in \theta_n$ if and only if $b - a \in n\mathbb{Z}$. Prove that θ_n is an equivalence relation.

Problem 3. Let n be a fixed positive integer and consider the equivalence relation θ_n .

Part (a). Suppose that $r, s \in \mathbb{Z}_n$, and suppose that $(r, s) \in \theta_n$. Explain why we must conclude $r = s$.

Part (b). If $a \in \mathbb{Z}$, explain why $a \in [r]_{\theta_n}$ for some $r \in \mathbb{Z}_n$.

Part (c). Use Parts (a) and (b) to explain why the equivalence relation θ_n partitions \mathbb{Z} into exactly n equivalence classes.

Theorem 10.1

Let $\mathcal{G} = (G, *)$ be any group, and let $H \subseteq G$ be nonempty. Define $\theta_H \subseteq G \times G$ by the rule $(a, b) \in \theta_H$ if and only if $b * a^{-1} \in H$. The following statements are logically equivalent.

1. The set H is a subgroup of \mathcal{G} .
2. The set θ_H is an equivalence relation on G .

Proof of Theorem 10.1

Problem 4. We will first prove that Claim 1 implies Claim 2.

Part (a). Let $u \in G$ and let ε be the identity for \mathcal{G} . Use the fact that $\varepsilon \in H$ to prove that $(u, u) \in \theta_H$.

Part (b). Suppose that $(x, y) \in \theta_H$. Use the fact that H is closed under the formation of inverses to prove that $(y, x) \in \theta_H$.

Part (c). Suppose that $(x, y), (y, z) \in \theta_H$. Use the fact that H is closed under the group operation to prove that $(x, z) \in \theta_H$.

Problem 5. Now, we will prove that Claim (2) implies Claim (1). Suppose the set θ_H is an equivalence relation on G , and let ε be the identity of \mathcal{G} .

Part (a). Use the fact that θ_H is reflexive to show that $\varepsilon \in H$.

Part (b). Suppose that $x \in H$. Use the fact that $\varepsilon \in H$ and the fact that θ_H is symmetric to show that $x^{-1} \in H$.

Part (c). Suppose that $x, y \in H$. Use Part (b) to explain why we know $(y^{-1}, \varepsilon) \in \theta_H$ and $(\varepsilon, x) \in \theta_H$.

Part (d). Use Part (c) and the fact that θ_H is transitive to show $x * y \in H$.

Problem 6. Let $\mathcal{G} = (G, *)$ be any group, and let $H \subseteq G$ be a subgroup of \mathcal{G} . Suppose that θ_H is defined as in Theorem 10.1, and let a be any fixed member of G . Show that the following statements are logically equivalent for any $y \in G$.

1. The element y is a member of $[a]_{\theta_H}$.
2. There exist some $x \in H$ such that $y = x * a$.

Problem 6 tells us that the equivalence classes modulo θ_H have a simple form. In particular, for all $a \in G$ we know

$$[a]_{\theta_H} = \{x * a : x \in H\}$$

Left and Right Cosets of a Subgroup

Let $\mathcal{X} = (X, *)$ be any group, and let $H \subseteq X$ be a subgroup of \mathcal{X} . For any $a \in X$, we let

$$a * H = \{a * x : x \in H\} \qquad H * a = \{x * a : x \in H\}$$

denote the *left coset* and *right coset*, respectively for the subgroup H determined by the element a .

In light of Problem 6, we see that the equivalence classes $[a]_{\theta_H}$ are precisely the right cosets of H .

Problem 7. Consider the group $\mathcal{S}_5 = (S_5, \circ)$ of permutations on the five-element set $X = \{1, 2, 3, 4, 5\}$ under function composition. Let

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix} \right\}$$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$$

Construct the left coset $\alpha \circ H$ and the right coset $H \circ \alpha$.

Theorem 10.2

Let $\mathcal{X} = (X, *)$ be any group, and suppose that H is a subgroup of \mathcal{X} . If we let

$$RC_H = \{H * a : a \in X\}$$

then the following statements are true:

1. Every element of X will appear in exactly one member of RC_H .
2. There is a bijection between $H * a$ and $H * b$ for all $a, b \in H$.

Proof of Theorem 10.2

Note that Statement 1 is true because the right cosets of H are precisely the equivalence classes modulo θ_H . Let's consider Statement 2.

Problem 8. Let $a, b \in X$ and consider the function $f: H * a \rightarrow H * b$ defined by $f(u) = (u * a^{-1}) * b$. Prove that the function f is one-to-one and onto.

Problem 9. Let $\mathcal{X} = (X, *)$ be any group, and suppose that H is a subgroup of \mathcal{X} . Based on the previous problems, decide whether or not the following statements are true. Justify your reasoning.

1. The subgroup H is always a member of RC_H .
2. If a appears in a particular right coset, it will never appear in any other right coset.

Problem 10. Consider the product group $\mathcal{Z}_6 \times \mathcal{Z}_2 \times \mathcal{Z}_2$ and the subgroup

$$H = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}$$

Construct the family RC_H of right cosets for the subgroup H .

$$\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0,0), (1,0,0), (2,0,0), (3,0,0), (4,0,0), (5,0,0), (0,1,0), (1,1,0), (2,1,0), (3,1,0), (4,1,0), (5,1,0), (0,1,1), (1,1,1), (2,1,1), (3,1,1), (4,1,1), (5,1,1)\}$$

Problem 10. Consider the group $\mathcal{S}_\times = (\mathcal{S}_\times, *)$ of symmetries on the cross and let $H = \{F, RRRR\}$.

Part (a). Construct the family RC_H of right cosets for the subgroup H .

Part (b). Construct the family LC_H of left cosets for the subgroup H .

Part (c). Are the members of LC_H the same as the members of RC_H ?

Theorem 10.3 (Lagrange's Theorem)

Let $\mathcal{X} = (X, *)$ be any finite group, and suppose that H is a subgroup of \mathcal{X} . The number of elements in H must be a divisor of the number of elements in X .

Proof of Theorem 10.3

Suppose that X contains exactly n elements, and form the equinumerous partition RC_H . Since H is closed under the operation $*$, we know that $H \in RC_H$.

[Why do we know this?]

Since X is finite, we know that RC_H contains only finitely many members. Suppose that RC_H contains exactly r members. Each member of RC_H must contain exactly the same number of elements. Let m be this number. It follows that $n = rm$.

[Why can we conclude this?]

We may conclude that m is a divisor of n . Since every member of RC_H contains exactly m elements, and since H is a member of RC_H , we have shown that the number of elements in H must be a divisor of the number of elements in X .

QED

Problem 11. Suppose that $\mathcal{X} = (X, *)$ is any finite group containing at least two elements, and suppose that $a \in X$ is not the identity. If the number of elements in X is *prime*, use Problem 6 from Investigation 9 and Lagrange's Theorem to prove that \mathcal{X} is cyclic with a as a generator.

Problem 12. Let $\mathcal{X} = (X, *)$ be any group, and let H be a subgroup of \mathcal{X} .

Part (a). For each $a \in X$, prove that there is a bijection between the left coset $a * H$ and the right coset $H * a$.

Part (b). If X is *finite*, explain why the families LC_H and RC_H must have the same number of members.

Index of a Subgroup in a Finite Group

Let $\mathcal{X} = (X, *)$ be any finite group, and suppose that H is a subgroup of \mathcal{X} . The number of members in LC_H (and in RC_H) is called the *index* of the subgroup H in the group \mathcal{X} . The index of H in the group \mathcal{X} is traditionally denoted by the symbol

$$[X|H]$$

Homework.

Problem 1. Consider the product group $\mathcal{Z}_6 \times \mathcal{Z}_2 \times \mathcal{Z}_2$ and the subgroup

$$H = \{(0,0,0), (3,0,0), (3,1,1), (0,1,1)\}$$

Part (a). How many distinct right cosets for H are there?

Part (b). Construct the right cosets for H .

Problem 2. Consider the group $\mathcal{S}_4 = (S_4, \circ)$ of permutations on the four-element set $X = \{1,2,3,4\}$ under function composition along with the subgroup

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \right\}$$

Part (a). How many distinct left cosets for H are there?

Part (b). Construct the left cosets for H .

Part (c). Construct the right cosets for H .

Problem 3. Let $\mathcal{X} = (X, *)$ be any group, let H be a subgroup of \mathcal{X} , and let $x, y \in X$. Prove $y \in x * H$ if and only if $y^{-1} * x \in H$.

The *quaternion group* is an important finite group of 2×2 matrices with complex entries. Let $i = \sqrt{-1}$ and let

$$\begin{aligned} \varepsilon &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & I &= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, & J &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & K &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\ -\varepsilon &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, & -I &= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, & -J &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, & -K &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \end{aligned}$$

If we let $Qua = \{\varepsilon, I, J, K, -\varepsilon, -I, -J, -K\}$, then Qua forms a group under (complex) matrix multiplication. The operation table for the quaternion group $Qua = (Qua, *)$ is shown below.

*	ε	$-\varepsilon$	I	$-I$	J	$-J$	K	$-K$
ε	ε	$-\varepsilon$	I	$-I$	J	$-J$	K	$-K$
$-\varepsilon$	$-\varepsilon$	ε	$-I$	I	$-J$	J	$-K$	K
I	I	$-I$	$-\varepsilon$	ε	K	$-K$	$-J$	J
$-I$	$-I$	I	ε	$-\varepsilon$	$-K$	K	J	$-J$
J	J	$-J$	$-K$	K	$-\varepsilon$	ε	I	$-I$
$-J$	$-J$	J	K	$-K$	ε	$-\varepsilon$	$-I$	I
K	K	$-K$	J	$-J$	$-I$	I	$-\varepsilon$	ε
$-K$	$-K$	K	$-J$	J	I	$-I$	ε	$-\varepsilon$

Problem 4. Is the Quaternion Group isomorphic to the group $S_x = (S_x, *)$ of symmetries on the cross? Justify your answer.

Problem 5. Consider the subgroup $H = \{\varepsilon, -\varepsilon\}$.

Part (a). Construct the set of left cosets for H .

Part (b). Construct the set of right cosets for H .

Problem 6. If X is any subgroup of the Quaternion Group, what are the possibilities for the number of elements X could contain?

Problem 7. Identify all of the subgroups of the Quaternion Group.

Problem 8. Suppose that $\mathcal{X} = (X, *)$ is any group, and suppose that H is any subgroup of \mathcal{X} . For all $a \in X$ prove that $H * a = H$ if and only if $a \in H$.