Let X be any nonempty set, and suppose that $\theta \subseteq X \times X$. Recall that θ is an *equivalence relation* on the set X provided the following conditions are met:

- The set θ is reflexive; that is, $(a, a) \in \theta$ for all $a \in X$.
- The set θ is symmetric; that is, *if* $(a, b) \in \theta$ *then* $(b, a) \in \theta$.
- The set θ is transitive; that is, *if* $(a, b), (b, c) \in \theta$ *then* $(a, c) \in \theta$.

Let X be any nonempty set, and suppose that $\theta \subseteq X \times X$ is an equivalence relation on X. For each $a \in X$, let

$$[a]_\theta = \{b \in X : (a, b) \in \theta\}$$

These sets are known as the *equivalence classes modulo* θ.

Problem 1. Let X be any nonempty set, and suppose that $\theta \subseteq X \times X$ is an equivalence relation on X. Use symmetry and transitivity to prove the following statement.

If there exist $x \in [a]_\theta \cap [b]_\theta$ *then* $[a]_\theta = [b]_\theta$.

The result in Problem 1 (along with the fact that θ is reflexive) tells us the equivalence classes modulo θ form a *partition* of the set X --- that is, every member of X must appear in *exactly one* equivalence class modulo θ.

Let n be a fixed positive integer. In Investigation 2, you showed that two integers a and b have the same remainder when divided by n if and only if $b - a \in n\mathbb{Z}$. In this and subsequent investigations, we will explore the implications of this fact for groups in general.

Problem 2. Let n be a fixed positive integer and define $\theta_n \subseteq \mathbb{Z} \times \mathbb{Z}$ by the rule $(a, b) \in \theta_n$ if and only if $b - a \in n\mathbb{Z}$. Prove that θ_n is an equivalence relation.

Problem 3. Let n be a fixed positive integer and consider the equivalence relation θ_n.

Part (a). Suppose that $r, s \in \mathbb{Z}_n$ and suppose that $r \neq s$. Explain why $[r]_{\theta_n} \neq [s]_{\theta_n}$.

Part (b). If $a \in \mathbb{Z}$, explain why $a \in [r]_{\theta_n}$ for some $r \in \mathbb{Z}_n$.

Theorem 10.1

Let $\mathcal{G} = (G,*)$ be any group, and let $H \subseteq G$ be nonempty. Define $\theta_H \subseteq G \times G$ by the rule $(a, b) \in \theta_H$ if and only if $b * a^{-1} \in H$. The following statements are logically equivalent.

1. The set H is a subgroup of $\mathcal{G}$.
2. The set θ_H is an equivalence relation on G.

Proof of Theorem 10.1

Problem 4. Adjust your argument from Problem 2 to prove that Statement 1 implies Statement 2.

Problem 5. Now, suppose the set θ_H is an equivalence relation on G. We need to show that H is a subgroup of $\mathcal{G}$. Let ε be the identity of $\mathcal{G}$.

Part (a). Use the fact that θ_H is reflexive to show that $\varepsilon \in H$.

Part (b). Suppose that $b \in H$. Use the fact that $\varepsilon \in H$ and the fact that θ_H is symmetric to show that $b^{-1} \in H$.

Part (c). Suppose that $a, b \in H$. Use the fact that $\varepsilon \in H$ and the fact that θ_H is transitive to show that $a * b \in H$.

Problem 6. Let $\boldsymbol{\mathcal{G}} = (G,*)$ be any group, and let $H \subseteq G$ be a subgroup of $\boldsymbol{\mathcal{G}}$. Suppose that θ_H is defined as in Theorem 10.1. Show that, for all $a \in G$, we have

$$[a]_{\theta_H} = \{x * a : x \in H\}$$

Left and Right Cosets of a Subgroup

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let $H \subseteq X$ be a subgroup of $\boldsymbol{\mathcal{X}}$. For any $a \in X$, we let

$$a * H = \{a * x : x \in H\} \qquad\qquad H * a = \{x * a : x \in H\}$$

denote the *left coset* and *right coset*, respectively for the subgroup H determined by the element a.

In light of Problem 6, we see that the equivalence classes $[a]_{\theta_H}$ are precisely the right cosets of H.

Problem 7. Consider the group $\boldsymbol{S}_5 = (S_5, \circ)$ of permutations on the five-element set $X = \{1,2,3,4,5\}$ under function composition. Let

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix} \right\}$$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$$

Construct the left coset $\alpha \circ H$ and the right coset $H \circ \alpha$.

Theorem 10.2

Let $\boldsymbol{X} = (X, *)$ be any group, and suppose that H is a subgroup of $\boldsymbol{X}$. If we let

$$RC_H = \{H * a : a \in X\}$$

then the following statements will always be true:

1. Every element of X will appear in exactly one member of RC_H.
2. There is a bijection between $H * a$ and $H * b$ for all $a, b \in H$.

Proof of Theorem 10.1

Note that Statement 1 is true because the right cosets of H are precisely the equivalence classes modulo θ_H. Let's consider Statement 2.

Problem 8. Let $a, b \in X$ and consider the function $f: H * a \to H * b$ defined by $f(u) = (u * a^{-1}) * b$. Prove that the function f is one-to-one and onto.

Theorem 10.2 gives us some very helpful tips to save time when constructing cosets.

Tips for Constructing Right Cosets

- The subgroup H is always a member of RC_H.
- If a appears in a particular right coset, it will never appear in any other left coset.

Problem 9. Consider the product group $\boldsymbol{\mathcal{Z}}_6 \times \boldsymbol{\mathcal{Z}}_2 \times \boldsymbol{\mathcal{Z}}_2$ and the subgroup

$$H = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}$$

Construct the family RC_H of right cosets for the subgroup H.

Problem 10. Consider the group $\boldsymbol{\mathcal{S}}_\times = (S_\times, *)$ of symmetries on the cross and let $= \{F, RRRR\}$.

Part (a). Construct the family RC_H of right cosets for the subgroup H.

Part (b). Construct the family LC_H of *left* cosets for the subgroup H.

Part (c). Are the members of LC_H the same as the members of RC_H?

Theorem 10.3 (Langrange's Theorem)

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any finite group, and suppose that H is a subgroup of $\boldsymbol{\mathcal{X}}$. The number of elements in H must be a divisor of the number of elements in X.

Proof of Theorem 10.2

Suppose that X contains exactly n elements, and form the equinumerous partition LC_H. Since H is closed under the operation $*$, we know that $H \in RC_H$.

[Why do we know this?]

Since X is finite, we know that RC_H contains only finitely many members. Suppose that LC_H contains exactly r members. Each member of RC_H must contain exactly the same number of elements. Let m be this number. It follows that $n = rm$.

[Why can we conclude this?]

We may conclude that m is a divisor of n. Since every member of RC_H contains exactly m elements, and since H is a member of RC_H, we have shown that the number of elements in H must be a divisor of the number of elements in X.

QED

Problem 11. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is any finite group containing at least two elements, and suppose that $a \in X$ is not the identity. If the number of elements in X is *prime*, use Problem 6 from Investigation 9 and Lagrange's Theorem to prove that is $\boldsymbol{\mathcal{X}}$ cyclic with a as a generator.

Problem 12. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, and let H be a subgroup of $\boldsymbol{\mathcal{X}}$.

Part (a). For each $a \in X$, prove that there is a bijection between the left coset $a * H$ and the right coset $H * a$.

Part (b). If X is *finite*, explain why the families LC_H and RC_H must have the same number of members.

Index of a Subgroup in a Finite Group

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any finite group, and suppose that H is a subgroup of $\boldsymbol{\mathcal{X}}$. The number of members in LC_H (and in RC_H) is called the *index* of the subgroup H in the group $\boldsymbol{\mathcal{X}}$. The index of H in the group $\boldsymbol{\mathcal{X}}$ is traditionally denoted by the symbol

$$[X|H]$$

Homework.

Problem 1. Consider the product group $\boldsymbol{\mathcal{Z}}_6 \times \boldsymbol{\mathcal{Z}}_2 \times \boldsymbol{\mathcal{Z}}_2$ and the subgroup

$$H = \{(0,0,0), (3,0,0), (3,1,1), (0,1,1)\}$$

Part (a). How many distinct right cosets for H are there?

Part (b). Construct the right cosets for H.

Problem 2. Consider the group $\boldsymbol{\mathcal{S}}_4 = (S_4,\circ)$ of permutations on the four-element set $X = \{1,2,3,4\}$ under function composition along with the subgroup

$$H = \left\{\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 1 & 2\end{pmatrix}\right\}$$

Part (a). How many distinct left cosets for H are there?

Part (b). Construct the left cosets for H.

Part (c). Construct the right cosets for H.

Problem 3. Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group, let H be a subgroup of $\boldsymbol{\mathcal{X}}$, and let $x, y \in X$. Prove $y \in x * H$ if and only if $y^{-1} * x \in H$.

The *quaternion group* is an important finite group of 2×2 matrices with complex entries. Let $i = \sqrt{-1}$ and let

$$\varepsilon = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad I = \begin{bmatrix}0 & i\\ i & 0\end{bmatrix}, \quad J = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \quad K = \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}$$

$$-\varepsilon = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, \quad -I = \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}, \quad -J = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \quad -K = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$$

If we let Qua $= \{\varepsilon, I, J, K, -\varepsilon, -I, -J, -K\}$, then Qua forms a group under (complex) matrix multiplication. The operation table for the quaternion group $\boldsymbol{Qua} = (\text{Qua}, *)$ is shown below.

$*$	$\boldsymbol{\varepsilon}$	$\boldsymbol{-\varepsilon}$	$\boldsymbol{I}$	$\boldsymbol{-I}$	$\boldsymbol{J}$	$\boldsymbol{-J}$	$\boldsymbol{K}$	$\boldsymbol{-K}$
$\boldsymbol{\varepsilon}$	ε	$-\varepsilon$	I	$-I$	J	$-J$	K	$-K$
$\boldsymbol{-\varepsilon}$	$-\varepsilon$	ε	$-I$	I	$-J$	J	$-K$	K
$\boldsymbol{I}$	I	$-I$	$-\varepsilon$	ε	K	$-K$	$-J$	J
$\boldsymbol{-I}$	$-I$	I	ε	$-\varepsilon$	$-K$	K	J	$-J$
$\boldsymbol{J}$	J	$-J$	$-K$	K	$-\varepsilon$	ε	I	$-I$
$\boldsymbol{-J}$	$-J$	J	K	$-K$	ε	$-\varepsilon$	$-I$	I
$\boldsymbol{K}$	K	$-K$	J	$-J$	$-I$	I	$-\varepsilon$	ε
$\boldsymbol{-K}$	$-K$	K	$-J$	J	I	$-I$	ε	$-\varepsilon$

Problem 4. Is the Quaternion Group isomorphic to the group $\boldsymbol{S}_\times = (S_\times, *)$ of symmetries on the cross? Justify your answer.

Problem 5. Consider the subgroup $H = \{\varepsilon, -\varepsilon\}$.

Part (a). Construct the set of left cosets for H.

Part (b). Construct the set of right cosets for H.

Problem 6. If X is any subgroup of the Quaternion Group, what are the possibilities for the number of elements X could contain?

Problem 7. Identify all of the subgroups of the Quaternion Group.

Problem 8. Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ is any group, and suppose that H is any subgroup of $\boldsymbol{\mathcal{X}}$. For all $a \in X$ prove that $H * a = H$ if and only if $a \in H$.