Let $X = \{a, b, c, d, e\}$ and consider the binary operations on X defined by the tables shown below.

$\bowtie$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$
$\boldsymbol{a}$	a	b	c	d	e
$\boldsymbol{b}$	b	a	d	e	c
$\boldsymbol{c}$	c	e	a	b	d
$\boldsymbol{d}$	d	c	e	a	b
$\boldsymbol{e}$	e	d	b	c	a

$\Cap$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$
$\boldsymbol{a}$	b	b	a	d	e
$\boldsymbol{b}$	a	b	c	d	e
$\boldsymbol{c}$	d	e	e	b	b
$\boldsymbol{d}$	d	c	a	a	b
$\boldsymbol{e}$	a	d	c	c	a

$\diamond$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$	$\boldsymbol{e}$
$\boldsymbol{a}$	b	d	e	a	c
$\boldsymbol{b}$	d	a	c	e	b
$\boldsymbol{c}$	c	e	a	b	d
$\boldsymbol{d}$	a	b	d	c	e
$\boldsymbol{e}$	e	c	b	d	a

Problem 1. Are any of these binary operations on the set X commutative? How did you decide?

Problem 2. Are any of these operations associative? How did you decide?

A set X endowed with operations is called an *algebra.* The set X is called the *universe* of the algebra. When we want to discuss a set X endowed with a particular binary operation *, we usually couple them together as an ordered pair $(X, *)$ and give this pair a name.

For example, we could let $\boldsymbol{\mathcal{X}}_1 = (X, \bowtie)$, $\boldsymbol{\mathcal{X}}_2 = (X, \Cap)$, and $\boldsymbol{\mathcal{X}}_3 = (X, \diamond)$ represent the three algebras defined by the tables above.

Problem 3. Consider the algebra $\boldsymbol{\mathcal{X}}_1$.

Part (a). Does the equation $m \bowtie x = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Part (b). Does the equation $x \bowtie m = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Problem 4. Consider the algebra $\boldsymbol{\mathcal{X}}_2$.

Part (a). Does the equation $m \Cap x = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Part (b). Does the equation $x \Cap m = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Problem 5. Consider the algebra $\boldsymbol{\mathcal{X}}_3$.

Part (a). Does the equation $m \diamond x = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Part (b). Does the equation $x \diamond m = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Sudoku Property

Suppose that X is a finite, nonempty set endowed with a binary operation *. We say that the algebra $\boldsymbol{\mathcal{X}} = (X,*)$ satisfies the *sudoku property* provided every member of X appears exactly once in each row and exactly once in each column of the operation table for $\boldsymbol{\mathcal{X}}$.

Problem 6. For the algebras $\boldsymbol{\mathcal{X}}_1$, $\boldsymbol{\mathcal{X}}_2$, and $\boldsymbol{\mathcal{X}}_3$ is there a connection between satisfying the Sudoku Property and the equations $m * x = n$ and $x * m = n$ having a unique solution for all $m, n \in X$? (Here * represents any one of the three operations.)

> ***Theorem 4.1***
>
> Suppose that X is a finite, nonempty set endowed with a binary operation *. The following statements are logically equivalent for the algebra $\boldsymbol{\mathcal{X}} = (X,*)$.
>
> 1. The algebra $\boldsymbol{\mathcal{X}}$ satisfies the Sudoku Property.
> 2. The equations $m * x = n$ and $x * m = n$ have a unique solution for all $m, n \in X$.

Proof of Theorem 4.1

Suppose that the algebra $\boldsymbol{\mathcal{X}}$ satisfies the Sudoku Property. Let $m, n \in X$ be fixed and consider the equation $m * x = n$. Think about the "m-Row" of the operation table. Since the algebra satisfies the Sudoku Property, we know that n must appear exactly once in this row. This means that there is exactly one column of the operation table whose intersection with the "m-Row" contains the element n. Let's call this particular column the "b-column." It follows that $m * b = n$; consequently, the equation $m * x = n$ has exactly one solution, namely $x = b$.

Consider the equation $x * m = n$. Think about the "m-Column" of the operation table. Since the algebra satisfies the Sudoku Property, we know that n must appear exactly once in this column. This means that there is exactly one row of the operation table whose intersection with the "m-Column" contains the element n. Let's call this particular column the "c-column." It follows that $c * m = n$; consequently, the equation $x * m = n$ has exactly one solution, namely $x = c$.

Conversely, suppose that the equations $m * x = n$ and $x * m = n$ have exactly one solution for all $m, n \in X$. First, consider the "m-Row" of the operation table. We need to prove that every element of X appears exactly once in this row. To this end, suppose $n \in X$. We know that the equation $m * x = n$ has exactly one solution. Suppose that $x = b$ is the unique solution to this equation. This tells us that the intersection of the "m-Row" and the "b-Column" contains the element n. Thus, we know that the element n appears in the "m-Row." Moreover, we know that there is exactly one column whose intersection with the "m-Row" contains the element n; therefore, n appears exactly once in the "m-Row."

Now, consider the "m-Column" of the operation table. We need to prove that every element of X appears exactly once in this column. To this end, suppose $n \in X$. We know that the equation $x * m = n$ has exactly one solution. Suppose that $x = c$ is the unique solution to this equation. This tells us that the intersection of the "m-Column" and the "c-Row" contains the element n. Thus, we know that the element n appears in the "m-Column." Moreover, we know that there is exactly one row whose intersection with the "m-Column" contains the element n; therefore, n appears exactly once in the "m-Column."

QED

Problem 8. Let X be any nonempty set, and suppose that $f: X \to X$ is a function.

Part (a). What does it mean to say that the function f is *one-to-one*?

Part (b). What does it mean to say the function f is *onto*?

Problem 7. In each of the algebras $\boldsymbol{\mathcal{X}}_1$, $\boldsymbol{\mathcal{X}}_2$, and $\boldsymbol{\mathcal{X}}_3$, consider the function $\lambda_d: X \to X$ defined by $\lambda_d(x) = d * x$. Is this function one-to-one in any of the algebras? Is this function onto in any of the algebras?

Problem 8. Consider the algebra $\boldsymbol{S}_\Delta = (S_\Delta, *)$ of symmetries for the equilateral triangle that you constructed in the previous investigation. Let $\rho_{FR}: S_\Delta \to S_\Delta$ be defined by $\rho_{FR}(x) = x * (FR)$.

Part (a). What is the value of $\rho_{FR}\big(F(RR)\big)$?

Part (b). What is the solution to the equation $x * FR = F$?

Part (c). What is the relationship between the equation $x * FR = y$ and the value of $\rho_{FR}(x)$ for any fixed $y \in S_\Delta$?

> ***Theorem 4.2***
>
> Suppose that X is *any* nonempty set endowed with a binary operation $*$. The following statements are logically equivalent for the algebra $\boldsymbol{\mathcal{X}} = (X, *)$.
>
> 1. The equations $m * x = n$ and $x * m = n$ have a unique solution for all $m, n \in X$.
> 2. For every $m \in X$, the functions $\lambda_m: X \to X$ and $\rho_m: X \to X$ defined by $\lambda_m(x) = m * x$ and $\rho_m(x) = x * m$ are both one-to-one and onto.

Proof of Theorem 4.2

First, suppose that $m * x = n$ and $x * m = n$ have exactly one solution for all $m, n \in X$. Consider the function λ_m. To see that this function is one-to-one, suppose that $\lambda_m(a) = \lambda_m(b)$ for some $a, b \in X$. This tells us that

$$m * a = \lambda_m(a) \quad \text{and} \quad m * b = \lambda_m(a)$$

Consequently, $x = a$ and $x = b$ are both solutions to the equation $m * x = \lambda_m(a)$. Since this equation has exactly one solution, we must conclude that $a = b$. Therefore, the function λ_m is one-to-one.

To see that this function is onto, let $n \in X$. We must show that there exist $b \in X$ such that $\lambda_m(b) = n$. Consider the equation $m * x = n$. By assumption, this equation has a solution. Let $x = b$ be the solution to this equation. It follows that

$$n = m * b = \lambda_m(b)$$

Therefore, the function λ_m is onto.

The proof that the function ρ_m is both one-to-one and onto is similar and will be left as an exercise.

Conversely, suppose that the functions λ_m and ρ_m are both one-to-one and onto. Let $m, n \in X$ and consider the equation $m * x = n$. Since the function λ_m is onto, we know there exist $b \in X$ such that $\lambda_m(b) = n$. It follows that $m * b = n$; hence, $x = b$ is one solution to the equation. Now, suppose that $x = a$ is also a solution to this equation. This means

$$n = m * a = \lambda_m(a) \qquad \text{and} \qquad n = m * b = \lambda_m(b)$$

Consequently, we know that $\lambda_m(a) = \lambda_m(b)$. Since λ_m is assumed to be one-to-one, we may conclude that $a = b$. Therefore, the equation $m * x = n$ has exactly one solution.

The proof that $x * m = n$ has exactly one solution for all $m, n \in X$ is similar and will be left as an exercise.

QED

Sudoku Algebras

Suppose that X is *any* nonempty set endowed with a binary operation *. We say that the algebra $\boldsymbol{X} = (X,*)$ is a *Sudoku algebra* provided the equations $m * x = n$ and $x * m = n$ have a unique solution for all $m, n \in X$.

HOMEWORK

Problem 1. Complete the proof of Theorem 4.2. In particular,

Part (a). Suppose that $x * m = n$ has a unique solution for all $m, n \in X$ and use this assumption to prove that the function ρ_m is both one-to-one and onto for every $m \in X$.

Part (b). Suppose that the function ρ_m is both one-to-one and onto for every $m \in X$ and use this assumption to prove that $x * m = n$ has a unique solution for all $m, n \in X$.

Problem 2. Suppose that $\boldsymbol{X} = (X,*)$ is any Sudoku algebra. Prove that the algebra satisfies the *cancellation laws*. That is, for all , $a, b, c \in X$ we have

- Left Cancellation: $a * b = a * c$ implies $b = c$
- Right Cancellation: $b * a = c * a$ implies $b = c$

Problem 3. Does the algebra $\boldsymbol{S}_\times = (S_\times,*)$ of symmetries for the cross form a Sudoku algebra? Justify your answer.

Problem 4. Let M_2 represent the family of all 2×2 matrices with real entries, and let $*$ represent the binary operation of *matrix multiplication* on this set. In symbols, let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}$$

Demonstrate by example that the following equation has multiple solutions in the algebra $\mathcal{M}_2 = (M_2,*)$.

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

Problem 5. Let $[\mathbb{Z} \to \mathbb{Z}]$ represent the family of all functions $f: \mathbb{Z} \to \mathbb{Z}$, and consider the binary operation of function composition on this family. Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$ and let $h: \mathbb{Z} \to \mathbb{Z}$ be defined by $h(x) = x$. Demonstrate by example that the equation $g \circ f = h$ has multiple solutions in the algebra $\mathfrak{F}_\mathbb{Z} = ([\mathbb{Z} \to \mathbb{Z}],\circ)$. (Hint: Consider piece-wise defined functions for g.)