Let $X = \{a, b, c, d, e\}$ and consider the binary operations on X defined by the tables shown below.

$\bowtie$	***a***	***b***	***c***	***d***	***e***
a	*a*	*b*	*c*	*d*	*e*
b	*b*	*a*	*d*	*e*	*c*
c	*c*	*e*	*a*	*b*	*d*
d	*d*	*c*	*e*	*a*	*b*
e	*e*	*d*	*b*	*c*	*a*

$\Cap$	***a***	***b***	***c***	***d***	***e***
a	*b*	*b*	*a*	*d*	*e*
b	*a*	*b*	*c*	*d*	*e*
c	*d*	*e*	*e*	*b*	*b*
d	*d*	*c*	*a*	*a*	*b*
e	*a*	*d*	*c*	*c*	*a*

$\diamond$	***a***	***b***	***c***	***d***	***e***
a	*b*	*d*	*e*	*a*	*c*
b	*d*	*a*	*c*	*e*	*b*
c	*c*	*e*	*a*	*b*	*d*
d	*a*	*b*	*d*	*c*	*e*
e	*e*	*c*	*b*	*d*	*a*

Problem 1. Are any of these binary operations on the set X commutative? How did you decide?

Problem 2. Are any of these operations associative? How did you decide?

A set X endowed with operations is called an *algebra*. The set X is called the *universe* of the algebra. When we want to discuss a set X endowed with a particular binary operation *, we usually couple them together as an ordered pair $(X,*)$ and give this pair a name.

For example, we could let $\boldsymbol{\mathcal{X}}_1 = (X, \bowtie)$, $\boldsymbol{\mathcal{X}}_2 = (X, \Cap)$, and $\boldsymbol{\mathcal{X}}_3 = (X, \diamond)$ represent the three algebras defined by the tables above.

Problem 3. Consider the algebra $\boldsymbol{\mathcal{X}}_1$.

Part (a). Does the equation $m \bowtie x = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Part (b). Does the equation $x \bowtie m = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Problem 4. Consider the algebra $\boldsymbol{\mathcal{X}}_2$.

Part (a). Does the equation $m \Cap x = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Part (b). Does the equation $x \Cap m = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Problem 5. Consider the algebra $\boldsymbol{\mathcal{X}}_3$.

Part (a). Does the equation $m \diamond x = n$ have a single solution for *every* $m, n \in X$? How did you decide?

Part (b). Does the equation $x \diamond m = n$ have a single solution for *every* $m, n \in X$? How did you decide?

> ***Sudoku Property***
>
> Suppose that X is a finite, nonempty set endowed with a binary operation *. We say that the algebra $\boldsymbol{\mathcal{X}} = (X,*)$ satisfies the *sudoku property* provided every member of X appears exactly once in each row and exactly once in each column of the operation table for $\boldsymbol{\mathcal{X}}$ (not counting the column and row headings).

Problem 6. For the algebras $\boldsymbol{\mathcal{X}}_1$, $\boldsymbol{\mathcal{X}}_2$, and $\boldsymbol{\mathcal{X}}_3$ is there a connection between satisfying the Sudoku Property and the equations $m * x = n$ and $x * m = n$ having a unique solution for all $m, n \in X$? (Here * represents any one of the three operations.)

Theorem 4.1

Suppose that X is a finite, nonempty set endowed with a binary operation *. The following statements are logically equivalent for the algebra $\boldsymbol{\mathcal{X}} = (X,*)$.

1. The algebra $\boldsymbol{\mathcal{X}}$ satisfies the Sudoku Property.
2. The equations $m * x = n$ and $x * m = n$ have a unique solution for all $m, n \in X$.

Proof of Theorem 4.1

Suppose that the algebra $\boldsymbol{\mathcal{X}}$ satisfies the Sudoku Property. Let $m, n \in X$ be fixed and consider the equation $m * x = n$.

Problem 7. Think about the "m-Row" of the operation table. Use the Sudoku Property to explain why there must exist exactly one $b \in X$ such that $m * b = n$. (Think about the columns in the operation table.)

Problem 8. How would you modify your argument in Problem 7 to demonstrate that $x * m = n$ has exactly one solution for every $m, n \in X$?

Problem 9. Conversely, suppose that the equations $m * x = n$ and $x * m = n$ have exactly one solution for all $m, n \in X$.

Part (a). First, consider the "m-Row" of the operation table. Explain why every element of X must appear exactly once in this row.

Part (b). Now, consider the "m-Column" of the operation table. How would you modify your argument in Part (a) to show that every element of X appears exactly once in this column?

Sudoku Algebras

Suppose that X is *any* nonempty set endowed with a binary operation *. We say that the algebra $\boldsymbol{\mathcal{X}} = (X,*)$ is a *Sudoku algebra* provided the equations $m * x = n$ and $x * m = n$ have a unique solution for all $m, n \in X$.

HOMEWORK

Problem 1. Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ is any Sudoku algebra. Prove that the algebra satisfies the *cancellation laws*. That is, for all , $a, b, c \in X$ we have

- Left Cancellation: $a * b = a * c$ implies $b = c$
- Right Cancellation: $b * a = c * a$ implies $b = c$

Problem 2. Does the algebra $\boldsymbol{\mathcal{S}}_\times = (S_\times,*)$ of symmetries for the cross form a Sudoku algebra? Justify your answer.

Problem 3. Let M_2 represent the family of all 2×2 matrices with real entries, and let $*$ represent the binary operation of *matrix multiplication* on this set. In symbols, let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix}$$

Demonstrate by example that the following equation has multiple solutions in the algebra $\mathcal{M}_2 = (M_2,*)$.

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

Problem 4. Let $[\mathbb{Z} \to \mathbb{Z}]$ represent the family of all functions $f: \mathbb{Z} \to \mathbb{Z}$, and consider the binary operation of function composition on this family. Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$ and let $h: \mathbb{Z} \to \mathbb{Z}$ be defined by $h(x) = x$.

Part (a). Prove that the function $g : \mathbb{Z} \to \mathbb{Z}$ defined by

$$g(x) = \begin{cases} x/2 & \text{if} \quad x \text{ is even} \\ 0 & \text{if} \quad x \text{ is odd} \end{cases}$$

is a solution to the equation $X \circ f = h$ in the algebra $\mathfrak{F}_\mathbb{Z} = ([\mathbb{Z} \to \mathbb{Z}],\circ)$.

Part (b). Construct a different solution to the equation $X \circ f = h$ in the algebra $\mathfrak{F}_\mathbb{Z} = ([\mathbb{Z} \to \mathbb{Z}],\circ)$. Be sure to prove that your function is indeed a solution to the equation.