

In this investigation, we will explore one way to create new groups (sometimes with new properties) from groups we already have. If X and Y are nonempty sets, remember that the *Cartesian products* formed from X and Y are the sets

$$X \times Y = \{(a, b) : a \in X, b \in Y\} \quad Y \times X = \{(b, a) : a \in X, b \in Y\}$$

Theorem 7.1 (Products of Groups)

Let $\mathcal{X} = (X, *)$ and $\mathcal{Y} = (Y, \diamond)$ be any groups. If we define a binary rule \otimes on the Cartesian product $X \times Y$ by

$$(a, b) \otimes (c, d) = (a * c, b \diamond d)$$

then $X \times Y$ forms a group under \otimes . We denote this group by $\mathcal{X} \times \mathcal{Y} = (X \times Y, \otimes)$ and call it the *product group of \mathcal{X} and \mathcal{Y}* .

Proof of Theorem 7.1

Problem 1. First, note that since $*$ and \diamond are binary operations on the sets X and Y , respectively, we know that for all $(a, b), (c, d) \in X \times Y$, we must have $a * c \in X$ and $b \diamond d \in Y$. Hence, $(a, b) \otimes (c, d) \in X \times Y$, and we may conclude that \otimes is a binary operation on $X \times Y$. Therefore, we know $\mathcal{X} \times \mathcal{Y}$ is an algebra.

Prove that the operation \otimes is associative.

Problem 2. Let ε_X and ε_Y denote the identity elements for \mathcal{X} and \mathcal{Y} , respectively. Prove that the algebra $\mathcal{X} \times \mathcal{Y}$ is a group.

Problem 3. Fill in the operation table for the product group $\mathcal{Z}_2 \times \mathcal{Z}_2$. (This group is called the *Klein Four-Group*.)

\otimes	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 0)				
(0, 1)				
(1, 0)				
(1, 1)				

Problem 4. Fill in the operation table for the product group $\mathcal{Z}_2 \times \mathcal{S}_\Delta$.

\otimes	(0, RRR)	(0, RR)	(0, R)	(0, F)	(0, FR)	(0, FRR)	(1, RRR)	(1, RR)	(1, R)	(1, F)	(1, FR)	(1, FRR)
(0, RRR)												
(0, RR)												
(0, R)												
(0, F)												
(0, FR)												
(0, FRR)												
(1, RRR)												
(1, RR)												
(1, R)												
(1, F)												
(1, FR)												
(1, FRR)												

Problem 5. Let $\mathcal{CR} = (CR, \circ)$ represent the cross ratio group introduced in Problem 2 of Investigation 5 and consider the product group $\mathcal{CR} \times \mathcal{Z}_4$.

Part (a). What is the inverse of the element $(u, 3)$ in the product group?

Part (b). Construct the powers $(r, 2)^{-1}$, $(r, 2)^{-2}$, $(r, 2)^{-3}$, $(r, 2)^{-4}$, $(r, 2)^{-5}$, and $(r, 2)^{-6}$.

Homework.

Problem 1. Construct the operation table for the product group $\mathcal{Z}_3 \times \mathcal{Z}_4$.

Problem 2. Construct the operation table for the product group $\mathcal{Z}_6 \times \mathcal{Z}_2$.

Problem 3. What is the inverse of the element $(2,3,4)$ in the product group $\mathcal{Z}_3 \times \mathcal{Z}_4 \times \mathcal{Z}_5$?

Problem 4. What is $(2,3,4)^{-7}$ in the product group $\mathcal{Z}_3 \times \mathcal{Z}_4 \times \mathcal{Z}_5$?

Cyclic Groups

Let $\mathcal{X} = (X,*)$ be any group. We say that \mathcal{X} is *cyclic* provided every member of X can be written as a power of some fixed element $a \in X$. The element a is called a *generator* for the group \mathcal{X} .

Problem 5. Consider the element $(2,3) \in \mathbb{Z}_3 \times \mathbb{Z}_4$.

Part (a). Compute the powers $(2,3)^1, (2,3)^2, \dots, (2,3)^{12}$ in the group $\mathcal{Z}_3 \times \mathcal{Z}_4$.

Part (b). Explain why $\mathcal{Z}_3 \times \mathcal{Z}_4$ is cyclic.

Problem 6. Is the Klein Four-Group a cyclic group? Explain.

Problem 7. Complete the following proof.

Let $\mathcal{X} = (X,*)$ be any group. If a is a generator for \mathcal{X} then a^{-1} is also a generator for \mathcal{X} .

Proof. Let $x \in X$. We need to show that there exists some integer n such that $x = (a^{-1})^n$. We have assumed that a is a generator for the group \mathcal{X} .

[Complete the argument.]