In this investigation, we will explore one way to create new groups (sometimes with new properties) from groups we already have. If X and Y are nonempty sets, remember that the *Cartesian products* formed from X and Y are the sets

$$X \times Y = \{(a, b) : a \in X, b \in Y\} \qquad Y \times X = \{(b, a) : a \in X, b \in Y\}$$

Problem 1. Construct the Cartesian products $\mathbb{Z}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times S_\Delta$.

Theorem 7.1 (Products of Groups)

Let $\boldsymbol{\mathcal{X}} = (X, *)$ and $\boldsymbol{\mathcal{Y}} = (Y, \diamond)$ be any groups. If we define a binary rule $\otimes$ on the Cartesian product $X \times Y$ by

$$(a, b) \otimes (c, d) = (a * c, b \diamond d)$$

then $X \times Y$ forms a group under $\otimes$. We denote this group by $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$ and call it the *product group of* $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$.

Proof of Theorem 7.1

First, note that since * and $\diamond$ are binary operations on the sets X and Y, respectively, we know that for all $(a, b), (c, d) \in X \times Y$, we must have $a * c \in X$ and $b \diamond d \in Y$. Hence, $(a, b) \otimes (c, d) \in X \times Y$, and we may conclude that $\otimes$ is a binary operation on $X \times Y$.

We need to show that the operation $\otimes$ is associative. We know that * and $\diamond$ are associative binary operations on the sets X and Y, respectively. Let $(a, b), (c, d), (u, v) \in X \times Y$ and observe

$$\begin{aligned}(a, b) \otimes [(c, d) \otimes (u, v)] &= (a, b) \otimes (c * u, d \diamond v) \\ &= (a * [c * u], b \diamond [d \diamond v]) \\ &= ([a * c] * u, [b \diamond d] \diamond v) \\ &= [(a, b) \otimes (c, d)] \otimes (u, v)\end{aligned}$$

If we let ε_X and ε_Y denote the identity elements for $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$, respectively, then it is easy to see that the pair $(\varepsilon_X, \varepsilon_Y)$ serves as the identity element for $X \times Y$ under the operation $\otimes$. Likewise, if we let a^{-1} denote the inverse of $a \in X$ and let b' denote the inverse of $b \in Y$, then it is easy to see that (a^{-1}, b') serves as the inverse for $(a, b) \in X \times Y$.

Consequently, we may conclude that $X \times Y$ forms a group under the operation $\otimes$.

QED

Problem 2. Fill in the operation table for the product group $\mathcal{Z}_2 \times \mathcal{Z}_2$. (This group is called the *Klein Four-Group*.)

$\otimes$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$				
$(0,1)$				
$(1,0)$				
$(1,1)$				

Problem 3. Fill in the operation table for the product group $\mathcal{Z}_2 \times \mathcal{S}_\Delta$.

$\otimes$	$(0,RRR)$	$(0,RR)$	$(0,R)$	$(0,F)$	$(0,FR)$	$(0,FRR)$	$(1,RRR)$	$(1,RR)$	$(1,R)$	$(1,F)$	$(1,FR)$	$(1,FRR)$
$(0,RRR)$												
$(0,RR)$												
$(0,R)$												
$(0,F)$												
$(0,FR)$												
$(0,FRR)$												
$(1,RRR)$												
$(1,RR)$												
$(1,R)$												
$(1,F)$												
$(1,FR)$												
$(1,FRR)$												

Problem 4. Let $\mathcal{CR} = (CR,\circ)$ represent the cross ratio group introduced in Problem 2 of Investigation 5 and consider the product group $\mathcal{CR} \times \mathcal{Z}_4$.

Part (a). What is the inverse of the element $(u,3)$ in the product group?

Part (b). Construct the powers $(r,2)^{-1}$, $(r,2)^{-2}$, $(r,2)^{-3}$, $(r,2)^{-4}$, $(r,2)^{-5}$, and $(r,2)^{-6}$.

Homework.

Problem 1. Construct the operation table for the product group $\mathcal{Z}_3 \times \mathcal{Z}_4$.

Problem 2. Construct the operation table for the product group $\mathcal{Z}_6 \times \mathcal{Z}_2$.

Problem 3. What is the inverse of the element (2,3,4) in the product group $\mathcal{Z}_3 \times \mathcal{Z}_4 \times \mathcal{Z}_5$?

Problem 4. What is $(2,3,4)^{-7}$ in the product group $\mathcal{Z}_3 \times \mathcal{Z}_4 \times \mathcal{Z}_5$?

Cyclic Groups

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group. We way that $\boldsymbol{\mathcal{X}}$ is *cyclic* provided every member of X can be written as a power of some fixed element $a \in X$. The element a is called a *generator* for the group $\boldsymbol{\mathcal{X}}$.

Problem 5. Consider the element $(2,3) \in \mathbb{Z}_3 \times \mathbb{Z}_4$.

Part (a). Compute the powers $(2,3)^1, (2,3)^2, \ldots, (2,3)^{12}$ in the group $\mathcal{Z}_3 \times \mathcal{Z}_4$.

Part (b). Explain why $\mathcal{Z}_3 \times \mathcal{Z}_4$ is cyclic.

Problem 6. Is the Klein Four-Group a cyclic group? Explain.

Problem 7. Complete the following proof.

Let $\boldsymbol{\mathcal{X}} = (X,*)$ be any group. If a is a generator for $\boldsymbol{\mathcal{X}}$ then a^{-1} is also a generator for $\boldsymbol{\mathcal{X}}$.

Proof. Let $x \in X$. We need to show that there exists some integer n such that $x = (a^{-1})^n$. We have assumed that a is a generator for the group $\boldsymbol{\mathcal{X}}$.

[Complete the argument.]