In mathematics, the term "isomorphic" is used to describe two mathematical entities that possess "identical structure." Consider, for example, the groups $\mathcal{Z}_3 \times \mathcal{Z}_2$ and $\mathcal{Z}_6$. The operation tables for these two groups are shown below.

$\otimes$	**(0, 0)**	**(0, 1)**	**(1, 0)**	**(1, 1)**	**(2, 0)**	**(2, 1)**
(0, 0)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(0, 1)	(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)
(1, 0)	(1,0)	(1,1)	(2,0)	(2,1)	(0,0)	(0,1)
(1, 1)	(1,1)	(1,0)	(2,1)	(2,0)	(0,1)	(0,0)
(2, 0)	(2,0)	(2,1)	(0,0)	(0,1)	(1,0)	(1,1)
(2, 1)	(2,1)	(2,0)	(0,1)	(0,0)	(1,1)	(1,0)

$\boxplus_6$	**0**	**3**	**4**	**1**	**2**	**5**
0	0	3	4	1	2	5
3	3	0	1	4	5	2
4	4	1	2	5	0	3
1	1	4	5	2	3	0
2	2	5	0	3	4	1
5	5	2	3	0	1	4

Compare the color patterns in the two operation tables, and you will see that the pattern presented in one table by a particular shaded element is identical to the pattern presented in the other table by the same-shaded element.

This tells us that we can arrange the elements of the group $\mathcal{Z}_6$ in a way that the operation table for this group looks exactly like the operation table for the group $\mathcal{Z}_3 \times \mathcal{Z}_2$. This means that the elements in the group $\mathcal{Z}_6$ are really just "renamed" versions of the elements in the group $\mathcal{Z}_3 \times \mathcal{Z}_2$. (In the diagram, the "renamed" elements have the same shade.)

Problem 1. The group of triangle symmetries is isomorphic to the cross-ratio group. The operation tables for these groups are shown below. In the table provided, rearrange the elements of the cross-ratio group so that the patterns presented in its table exactly match the patterns in the symmetries table.

$*$	RRR	RR	R	F	FR	FRR
RRR	RRR	RR	R	F	FR	FRR
RR	RR	R	RRR	FR	FRR	F
R	R	RRR	RR	FRR	F	FR
F	F	FRR	FR	RRR	R	RR
FR	FR	F	FRR	RR	RRR	R
FRR	FRR	FR	F	R	RR	RRR

$\circ$	ε	q	r	s	t	u
ε	ε	q	r	s	t	u
q	q	ε	u	t	s	r
r	r	s	ε	q	u	t
s	s	r	t	u	q	ε
t	t	u	s	r	ε	q
u	u	t	q	ε	r	s

$\circ$						

Problem 2. Do you think that the group of triangle symmetries is isomorphic to the group $\boldsymbol{\mathcal{Z}}_6$? Explain your reasoning.

Problem 3. Do you think that the Klein Four-Group $\boldsymbol{\mathcal{Z}}_2 \times \boldsymbol{\mathcal{Z}}_2$ is isomorphic to the group $\boldsymbol{\mathcal{Z}}_4$? Explain your reasoning.

It is usually impractical (or impossible) to attempt rearranging the operation table of one group to determine whether that group is isomorphic to another group. If we look carefully at how rearranging actually relates the members of each group, we can devise a more efficient test.

Suppose we are comparing two finite groups $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$. If one universe contains more elements than the other, then there is no way we could ever rearrange the operation table for the group $\boldsymbol{\mathcal{Y}}$ so that it matches the operation table for the group $\boldsymbol{\mathcal{X}}$.

- If two groups have any chance of being isomorphic, there must be a bijection (one-to-one and onto function) between their universes.

Suppose that X and Y contain the same number of elements. Every arrangement of the elements in the operation table for $\boldsymbol{\mathcal{Y}}$ creates a bijection between X and Y. For example, consider the arrangement of the elements of the group $\boldsymbol{\mathcal{Z}}_6$ appearing in the table at the beginning of this investigation. We can create a bijection $f\colon \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ by matching the shaded elements.

$$(0,0) \xrightarrow{f} 0 \quad (0,1) \xrightarrow{f} 3 \quad (1,0) \xrightarrow{f} 4 \quad (1,1) \xrightarrow{f} 1 \quad (2,0) \xrightarrow{f} 2 \quad (2,1) \xrightarrow{f} 5$$

This is only one of many bijections we could create between the sets $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$. (There are actually 720 of them.) However, this particular bijection has a special property. This bijection makes the patterns in the table for $\boldsymbol{\mathcal{Z}}_6$ exactly match the patterns in the table for $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_2$. This pattern matching can only occur if the following statement is true for all $a, b \in \mathbb{Z}_3 \times \mathbb{Z}_2$:

$$a \otimes b = c \Leftrightarrow f(a) \boxplus_6 f(b) = f(c)$$

Operation Preserving Functions

Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups, and suppose that $h\colon X \to Y$ is a function. We say that the function h *preserves the operation* provided $h(a * b) = h(a) \odot h(b)$ for all $a, b \in X$.

An operation preserving function $h\colon X \to Y$ is called a *homomorphism* from the group $\boldsymbol{\mathcal{X}}$ to the group $\boldsymbol{\mathcal{Y}}$. A homomorphism that is also a bijection is called an *isomorphism* between the groups $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Y}}$.

Problem 4. The bijection $g: \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ defined by the arrangement below is *not* an isomorphism between the groups $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_2$ and $\boldsymbol{\mathcal{Z}}_6$. What goes wrong?

$$(0,0) \xrightarrow{g} 0 \quad (0,1) \xrightarrow{g} 3 \quad (1,0) \xrightarrow{g} 2 \quad (1,1) \xrightarrow{g} 1 \quad (2,0) \xrightarrow{g} 4 \quad (2,1) \xrightarrow{g} 5$$

Problem 5. Show that the function $h: \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ defined by the assignment below is a homomorphism from $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_2$ to $\boldsymbol{\mathcal{Z}}_6$.

$$(0,0) \xrightarrow{h} 0 \quad (0,1) \xrightarrow{h} 0 \quad (1,0) \xrightarrow{h} 2 \quad (1,1) \xrightarrow{h} 2 \quad (2,0) \xrightarrow{h} 4 \quad (2,1) \xrightarrow{h} 4$$

Theorem 8.1 (Preservation of the Identity and Inverses)

Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups, and suppose that $h: X \to Y$ is a homomorphism. The following statements are true.

1. If ε is the identity for the group $\boldsymbol{\mathcal{X}}$, then $h(\varepsilon)$ is the identity for the group $\boldsymbol{\mathcal{Y}}$.
2. For all $a \in X$, we have $h(a^{-1})$ serving as the inverse for $h(a)$ in the group $\boldsymbol{\mathcal{Y}}$.

Proof of Theorem 8.1

Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups, and suppose that $h: X \to Y$ is a homomorphism. Also, suppose that δ is the identity for the group $\boldsymbol{\mathcal{Y}}$, and suppose that ε is the identity for the group $\boldsymbol{\mathcal{X}}$.

Problem 6. Use the fact that $\varepsilon * \varepsilon = \varepsilon$ to help you prove $h(\varepsilon) = \delta$.

Problem 7. Let $a \in X$. Use the fact that $h(\varepsilon) = h(a * a^{-1})$ to help you prove that $h(a^{-1})$ serves as the inverse for $h(a)$ in the group $\boldsymbol{\mathcal{Y}}$.

Theorem 8.1 gives us helpful hints when we are trying to construct an isomorphism between two groups. In particular, if $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are groups and $f : X \to Y$ is a bijection, then the following must be true for f to be an isomorphism:

1. The function f must assign the identity of $\boldsymbol{\mathcal{X}}$ to the identity of $\boldsymbol{\mathcal{Y}}$.
2. If a and b are inverses in the group $\boldsymbol{\mathcal{X}}$, then $f(a)$ and $f(b)$ must be inverses in the group $\boldsymbol{\mathcal{Y}}$.

These guidelines help us eliminate many bijections as possible isomorphisms. However, a bijection which satisfies both of these properties may *still* not be an isomorphism. (The function appearing in Problem 4 is an example.) In the end, we must prove that a bijection preserves the operation in order to guarantee that it serves as an isomorphism.

Problem 8. Find a different isomorphism between the groups $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_2$ and $\boldsymbol{\mathcal{Z}}_6$.

Problem 9. Prove the following result.

Suppose that $\boldsymbol{\mathcal{X}} = (X,*)$ *and* $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ *are groups. If* $f : X \to Y$ *is an isomorphism, then its inverse function* $g : Y \to X$ *is also an isomorphism.*

The previous result tells us that when trying to prove two groups $\boldsymbol{\mathcal{X}} = (X,*)$ and $\boldsymbol{\mathcal{Y}} = (Y,\odot)$ are isomorphic, we can consider bijections $f : X \to Y$ or bijections $g : Y \to X$.

Problem 10. Let $\mathbb{R}^*$ denote the set of nonzero real numbers, and consider the binary rule $\sqcap$ defined by

$$a \sqcap b = \frac{ab}{4}$$

In Problem 1 of Investigation 6, you showed that $\boldsymbol{\mathcal{R}}_{\sqcap} = (\mathbb{R}^*, \sqcap)$ is a group. Let $\boldsymbol{\mathcal{R}}_{\cdot} = (\mathbb{R}^*, \cdot)$ represent the group of nonzero real numbers under multiplication. Prove that the function $f : \mathbb{R}^* \to \mathbb{R}^*$ defined by

$$f(x) = 4 \cdot x$$

is an isomorphism between these groups.

Problem 11. Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ and $\boldsymbol{\mathcal{Y}} = (Y, \odot)$ are isomorphic groups. If $\boldsymbol{\mathcal{X}}$ is a commutative group, prove that $\boldsymbol{\mathcal{Y}}$ is also a commutative group.

Homework.

Problem 1. Consider the group $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ of integers under addition. For any fixed integer n, show that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = nx$ is a homomorphism from $\boldsymbol{\mathcal{Z}}$ to $\boldsymbol{\mathcal{Z}}$.

Problem 2. Consider the group $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ and the group $\boldsymbol{\mathcal{Z}}_n = (\mathbb{Z}_n, \boxplus_n)$. For each integer a, let r be the unique remainder obtained when a is divided by n. (See Investigation 2.) Complete the proof of the following statement.

The function $f : \mathbb{Z} \to \mathbb{Z}_n$ defined by $f(a) = r$ is a homomorphism from $\boldsymbol{\mathcal{Z}}$ to $\boldsymbol{\mathcal{Z}}_n$.

Proof. The Division Algorithm tells us that $0 \le f(a) < n$; hence we know that f is indeed a mapping from $\mathbb{Z}$ to $\mathbb{Z}_n$. Let $a, b \in \mathbb{Z}$. We need to prove that $f(a+b) = f(a) \boxplus_n f(b)$. The Division Algorithm tells us there exist unique integers m, p, q and r, s, t such that $0 \le r, s, t < n$ and

$$a + b = mn + r \qquad a = qn + s \qquad b = pn + t$$

We must show that $r = s \boxplus_n t$. [Why is this what we need to show?]

There exist unique integers j, k such that $0 \le k < n$ and $s + t = jn + k$, and we know that $s \boxplus_n t = k$. [Why do we know this?]

Now, consider the sum $a + b$.

[Fill this in]

Therefore, we know that $a + b = (q + p + j)n + k$, and we must conclude $k = r$. [Why must we conclude this?]

QED

Problem 3. Let $\boldsymbol{\mathcal{R}}^{\boldsymbol{p}} = (\mathbb{R}^+, \cdot)$ represent the group of positive real numbers under multiplication, and let $\boldsymbol{\mathcal{R}}_+ = (\mathbb{R}, +)$ denote the group of real numbers under addition. Prove that the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \ln(x)$ is an isomorphism between $\boldsymbol{\mathcal{R}}_+$ and $\boldsymbol{\mathcal{R}}^{\boldsymbol{p}}$.

Problem 4. Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ and $\boldsymbol{\mathcal{Y}} = (Y, \odot)$ are isomorphic groups, and suppose that $f : X \to Y$ is any isomorphism. Use the method of induction to prove the following statement.

If $a \in X$, then $f(a^n) = [f(a)]^n$ for every positive integer n.

Problem 5. Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ and $\boldsymbol{\mathcal{Y}} = (Y, \odot)$ are isomorphic groups, and suppose that $f : X \to Y$ is any isomorphism. Use the previous problem and the fact that f preserves inverses to prove the following statement.

If $a \in X$, then $f(a^n) = [f(a)]^n$ for every integer n.

Problem 6. Suppose that $\boldsymbol{\mathcal{X}} = (X, *)$ and $\boldsymbol{\mathcal{Y}} = (Y, \odot)$ are isomorphic groups, and suppose that $f : X \to Y$ is any isomorphism. Prove the following statement.

If is $\boldsymbol{\mathcal{X}}$ cyclic with generator a, then $\boldsymbol{\mathcal{Y}}$ is also cyclic with generator $f(a)$.