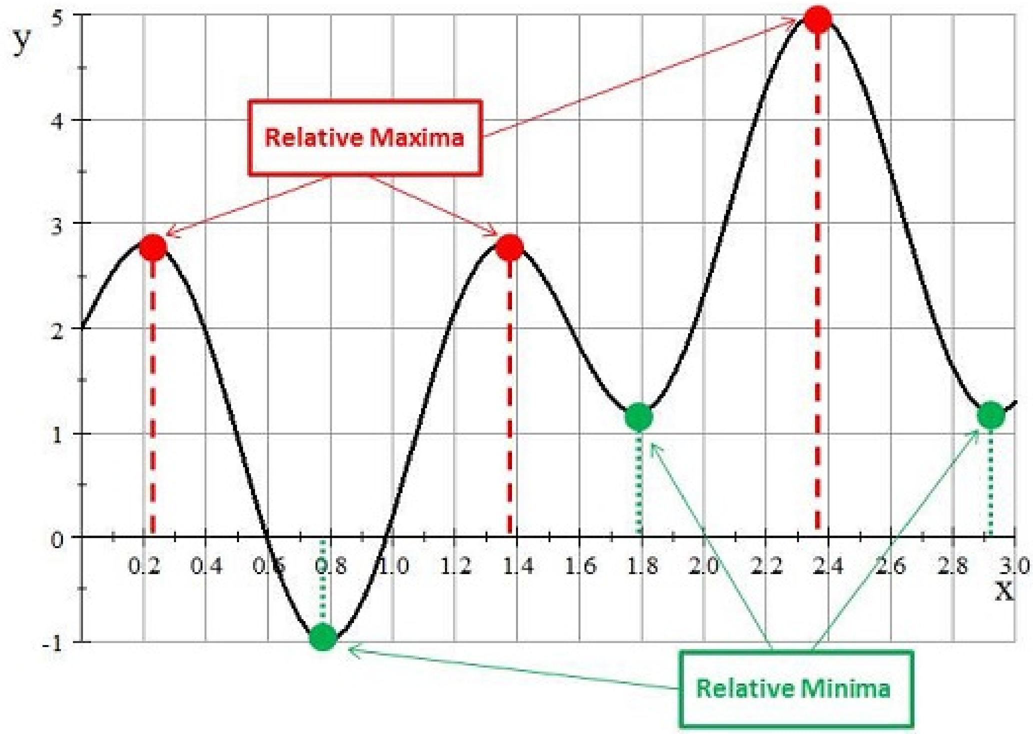
Absolute Maximum and Minimum Values for Functions

Back in Chapter 2, we introduced the notion of *relative* maximum and minimum outputs for a function. A function $y = f(x)$ is said to have a *relative maximum output* at an input value $x = a$ provided $f(a)$ is the largest output for f on some small input interval containing $x = a$. We say that f has a *relative minimum output* at an input value $x = b$ provided $f(b)$ is the smallest output on some input interval containing $x = b$.



We use the term *relative extrema* when we want to talk about the relative maximum or minimum outputs of a function. Relative extrema are a feature of the graph for f and as such do not change. There is another important form of extreme output for a function that depends on the specified input interval we want to consider.

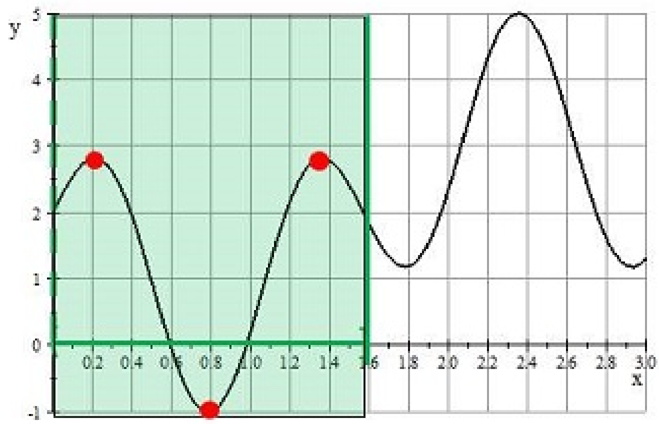
Let $y = f(x)$ be a function, and suppose that I is a subset of the domain for the function f.

- We say that f has an *absolute maximum output* for some input value $x = a$ in the set I provided $f(a) \geq f(x)$ for all input values x taken from the set I.
- We say that f has an *absolute minimum output* for some input value $x = a$ in the set I provided $f(a) \leq f(x)$ for all input values x taken from the set I.

Example 1 *Consider the function f whose graph is shown above. Where does f have its absolute maximum and absolute minimum outputs on the following specified sets?*

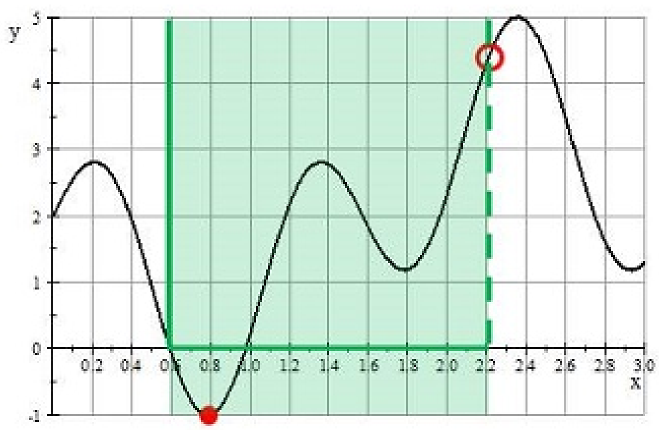
1. *The set* $0 < x \leq 1.6$
2. *The set* $0.6 \leq x < 2.2$
3. *The set* $1.0 \leq x \leq 2.2$

Solution. We will consider each set individually. First, consider the set $0 < x \leq 1.6$. The diagram below shows the graph of f along with this set.



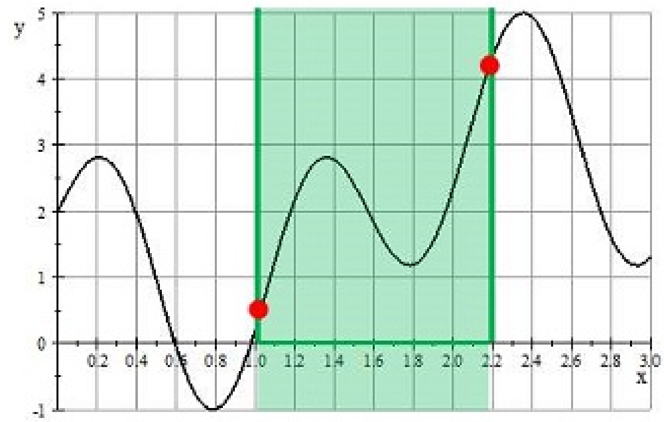
We can see from the diagram that the function f will have its absolute minimum output at approximately $x = 0.8$. The function will have its absolute maximum output at *two* input values — namely $x = 0.2$ and $x = 1.35$.

Now, consider the set $0.6 \leq x < 2.2$. The diagram below shows the graph of f along with this set.



We can see from the diagram that the function f will have its absolute minimum output as approximately $x = 0.8$. Now, the function f would have its maximum output at the right endpoint $x = 2.2$ *if this endpoint were part of the set.* However, since $x = 2.2$ is not part of the set, we conclude that f *has no absolute maximum output on this set.*

Finally, consider the set $1.1 \leq x \leq 2.2$. The diagram below shows the graph of f along with this set.



We can see from the diagram that the function f will have it absolute maximum and minimum outputs at the endpoints of this set. In particular, the absolute maximum output for f will occur at $x = 2.2$ and the absolute minimum output will occur at $x = 1.0$.

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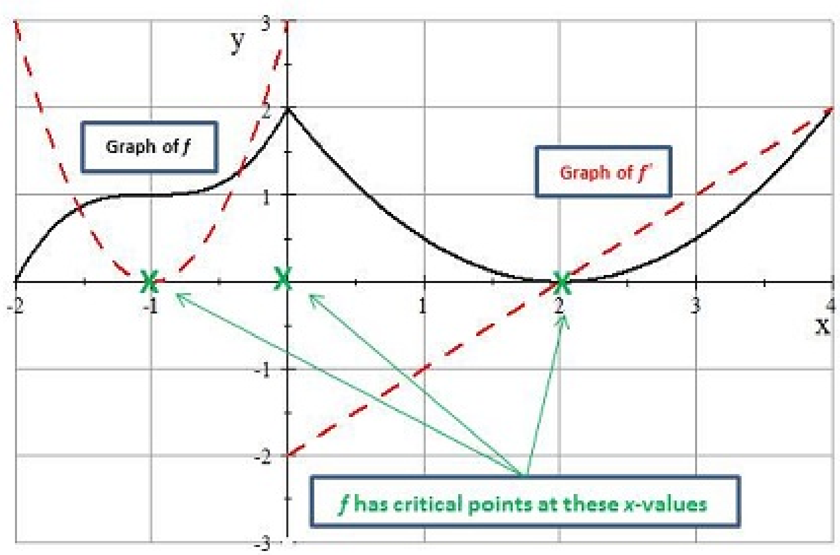
Notice that the relative extrema for f do not change in this example. They are a fixed part of the graph. However, sometimes the relative extrema serve as absolute extrema and sometimes they do not. That depends entirely on the set of input values we are considering.

We say that a function f has a *critical number* at some input value $x = a$ provided $f(a)$ exists and one of the following statements is true.

- The derivative function f' is NOT defined at $x = a$.
- The derivative function f' intersects the x-axis at $x = a$ (that is, $f'(a) = 0$).

Every relative extremum for a function f will is a critical number for f, however, not every critical point for f will be a relative extremum for f.



There is a simple procedure for identifying where a function f will have its absolute extreme values on a specified interval when the function is defined by a formula.

1. First, compute the derivative function for f.
2. Next, identify the critical numbers for f.
3. Finally, evaluate the function f at the critical numbers *that are in the specified set* along with the endpoints of the specified set, and compare.

Example 2 *Determine where the absolute extrema occur for the function* $f(x) = 2x-(x-1)^3$ *in the specified set* $-1 \le x < 5$.

Solution. First, observe that

$$f'(x) = 2 - 3(x-1)^2$$

Since there is no division in the formula for the derivative function, there will not be any input values where f' is undefined. Setting $f'(x) = 0$ tells us

$$\begin{aligned} 2 - 3(x-1)^2 = 0 \quad &\Longrightarrow \quad (x-1)^2 = \frac{2}{3} \\ &\Longrightarrow \quad x - 1 = \pm\sqrt{\frac{2}{3}} \\ &\Longrightarrow \quad x = 1 \pm \sqrt{\frac{2}{3}} \end{aligned}$$

Therefore, f has two critical numbers, namely $x \approx 0.1835$ and $x \approx 1.8165$. Finally, we compare the output of f for $x = -1$, $x = 5$, $x \approx 0.1835$, and $x \approx 1.8165$. Observe

$$f(-1) = 6 \qquad f(0.1835) \approx 0.91134 \qquad f(1.18165) \approx 2.3573 \qquad f(5) = -54$$

The largest output for f occurs when $x = -1$; therefore, f has an absolute maximum output at this input value. The smallest output for f occurs when $x = -5$. However, this endpoint is NOT in the specified set. Consequently, we conclude that f *has no absolute minimum output on this set.*

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Problem 1. Consider the function $f(x) = x^2 - 3(x-1)^2$.

Part (a): Differentiate the function f, and simplify your answer as much as possible.

Part (b): There is one critical number for the function f. Use Part (a) to identify it.

Part (c): Determine where the absolute extrema for f occur on the set $-1 \leq x \leq 3$.

Problem 2. Consider the function $f(x) = x - \sqrt{x}$.

Part (a): Differentiate the function f, and write your answer as a single fraction.

Part (b): There are two critical numbers for the function f. Use Part (a) to identify both.

Part (c): Determine where the absolute extrema for f occur on the set $0 < x < 2$.

HOMEWORK: Section 4.1, Pages 283-284 Problems 29, 31, 35, 37, 47, 49, 51, 52, 53, 54, 56