ALGEBRA REVIEW UNIT 5 ---- Common Factors

In this unit, we look at the distributive rule from a different perspective.

Suppose A is a sum of expressions. We say that an expression B is a *common factor* of A provided all the terms in A are like terms with respect to B.

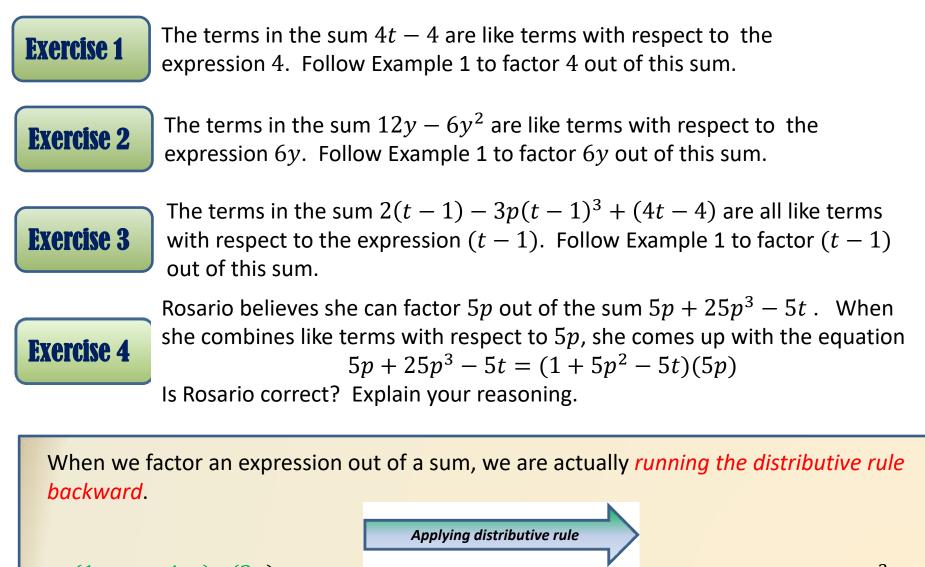
If we combine all the like terms with respect to *B*, we say that we have *factored B out of the terms*.

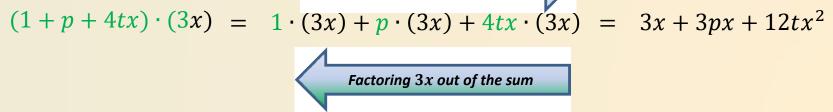
EXAMPLE 1. All of the terms in the sum $3x + 3px + 12tx^2$ are like terms with respect to the expression 3x. If we combine these like terms, we have

$$3x + 3px + 12tx^{2} = 1 \cdot (3x) + p \cdot (3x) + 4tx \cdot (3x)$$

 $= (1+p+4tx) \cdot (3x)$

By combining the like terms, we have factored 3x out of the expression.





Apply the distributive rule to the right-hand side of each equation to see if the proposed factoring is valid.

Part (a)
$$4pt^2 - 12xpt + 4pt \stackrel{?}{=} (t - 3x + 1)(4pt)$$

Part (b)
 $4x - 5y + 8xyt \stackrel{?}{=} (1 - 5y + 2yt)(4x)$

Part (c)
 $\frac{t}{px} - \frac{2r}{x} \stackrel{?}{=} (\frac{t}{p} - 2r) \cdot \frac{1}{x}$

EXAMPLE 2. Consider the quotient $\frac{3xy^2-4y}{xy}$.

Exercise 5

Notice that *y* is a factor of the denominator and a common factor of the numerator. This fact will allow us to simplify the quotient.

$$\frac{3xy^2 - 4y}{xy} = \frac{(3xy - 4)y}{xy} \qquad (Factor out y from the numerator)$$
$$= \frac{3xy - 4}{x} \cdot \frac{y}{y} \qquad (Apply MR2.)$$
$$= \frac{3xy - 4}{x} \cdot 1 \quad (As \ long \ as \ y \neq 0.) \qquad (Apply RR1 \ and \ exclude \ y = 0.)$$
$$= \frac{3xy - 4}{x} \qquad (As \ long \ as \ y \neq 0.)$$

In the previous example, we had to exclude y = 0 because the expression

 $\frac{3xy-4}{x}$

is defined when y = 0, but the expression

$$\frac{3xy^2 - 4y}{xy}$$

is NOT defined when y = 0. These two expressions will be equal for all values of y EXCEPT for y = 0. Note that *both* expressions are undefined when x = 0, so we don't have to exclude that possibility. (It is alright to do so, however.)

It is important to note that

$$\frac{3xy-4}{x} \neq 3y-4$$

for every nonzero value of x and y. For example, if we let x = 2 and y = 3, we have

$$\frac{3(2)(3) - 4}{2} = 7 \quad BUT \qquad 3(3) - 4 = 5$$



Use the value t = 6 to show that t

$$\frac{t}{t+1} \neq \frac{1}{1+1}$$

Exercise 7 Simplify the quotient $\frac{8t}{24y-8}$ by following Example 2.

Exercise 8

Simplify the quotient $\frac{u^2}{6u^4-u^2}$ by following Example 2. You will need to exclude a value of u.

Exercise 9 Simplify the quotient $\frac{6pt-18py}{12p-18py}$ by following Example 2. You will need to exclude a value of p.

Exercise 10

Simplify the quotient $\frac{3(t-2)^2}{8t-16}$. You will need to exclude a value of t.

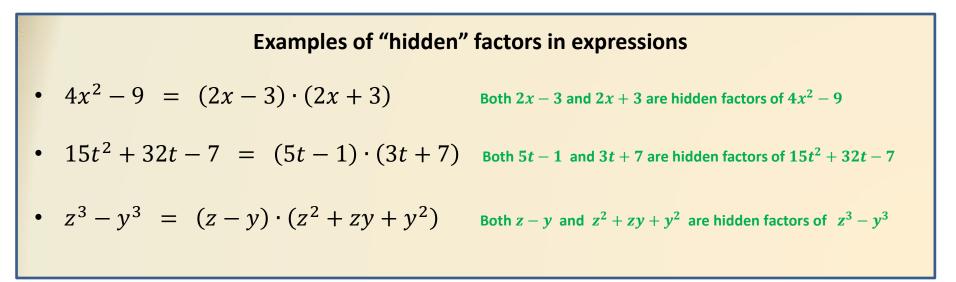
Exercise 11 Simplify the quotient $\frac{21x^3+3x}{12x^2}$. Explain why you do not have to exclude any values of x.



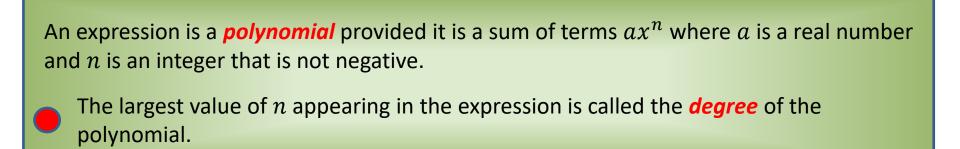
Use a specific value of r to show that the "simplification" below is not valid.

$$\frac{3r^2-7}{r+5} \quad \stackrel{?}{=} \quad \frac{3r-7}{1+5}$$

It is not always obvious that one expression is a factor of another. You are already familiar with many such examples, although you may not be aware of it.



The equations above all represent "factored polynomials," and you probably spent a lot of time in high school learning techniques for this kind of factoring.



In practice, you seldom need to factor a polynomial "from scratch" as the techniques you learned show you how to do. More often, you need to determine whether a given expression serves as a factor for a polynomial.

Determining whether a given expression is a factor of a polynomial is actually fairly easy. The method of "long division" provides the way.

EXAMPLE 3. Use long division to show that 3t + 7 is a factor of $15t^2 + 32t - 7$.

STEP 1A. Ask yourself "*What should I multiply 3t by in order to obtain* $15t^2$? ANSWER: I need to multiply 3t by 5t.

STEP 1B. Multiply the *entire* expression 3t + 7 by 5t and then *subtract* the result from $15t^2 + 32t - 7$

 $5t \leftarrow Your answer to STEP 1A.$ $3t + 7) \overline{15t^2 + 32t - 7}$ $-(\underline{15t^2 + 35t}) \leftarrow Subtract 5t \cdot (3t + 7) from 15t^2 + 32t - 7$ -3t - 7

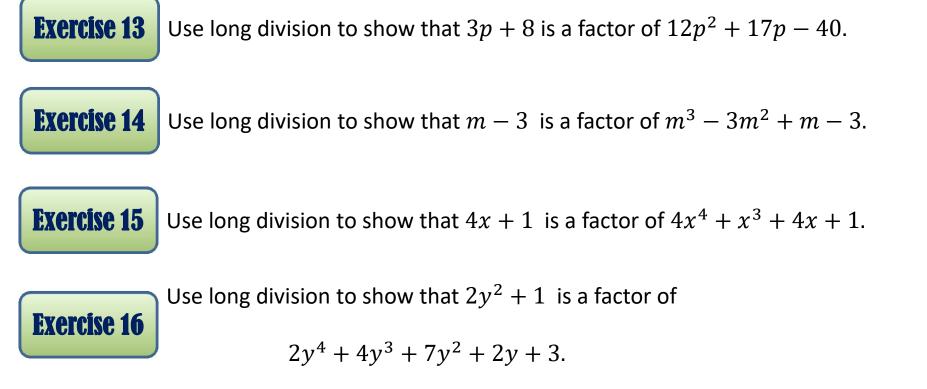
EXAMPLE 3 Continued

STEP 2A. Ask yourself "*What should I multiply* 3*t* by in order to obtain -3t? ANSWER: I need to multiply 3*t* by -1.

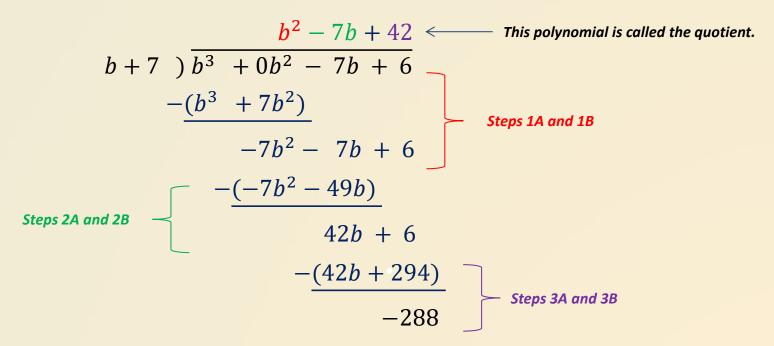
STEP 2B. Multiply the *entire* expression 3t + 7 by -1 and then *subtract* the result from -3t - 7

$$5t - 7 \qquad \text{Your answer to STEP 2A.} 3t + 7) \overline{15t^2 + 32t - 7} -(\underline{15t^2 + 35t}) -3t - 7 -(\underline{-3t - 7}) \qquad \text{Subtract } (-1) \cdot (3t + 7) \text{ from } -3t - 7 0$$

Since the process ends with 0, we know that 3t + 7 is indeed a factor of $15t^2 + 32t - 7$; in fact, the process tells us that $15t^2 + 32t - 7 = (3t + 7) \cdot (5t - 7)$.



Long division can also tell you when a given expression is NOT a factor of a polynomial. During the process, if you obtain a polynomial whose degree is SMALLER than the degree of the expression you are considering, the process stops. The expression you are considering is NOT a factor of the polynomial. **EXAMPLE 4.** Show that b + 7 is NOT a factor of $b^3 - 7b + 6$.



In Step 3B, we obtain -288. This polynomial has Degree 0, while b + 7 has Degree 1. Therefore, the process of long division stops; and we may conclude that b + 7 is NOT a factor of $b^3 - 7b + 6$. The last polynomial obtained in the long division process is called the *remainder*.

The process actually tells us more than this. We now know that

 $b^3 - 7b + 6 = (b + 7) \cdot (b^2 - 7b + 42) - 288$

Exercise 17

Show that 2r - 1 is NOT a factor of $12r^2 - 4$. Use your work to write $12r^2 - 4$ as the polynomial 2r - 1 times a quotient plus a remainder like in Example 4.

Exercise 18

Show that $t^2 + 3$ is NOT a factor of $t^3 + 2t + 8$. Use your work to write $t^3 + 2t + 8$ as the polynomial $t^2 + 3$ times a quotient plus a remainder.



Consider the quotient $\frac{2a^2-7a+6}{4a^2-13a+10}$.

Part (a)

Use long division to show that a - 2 is a factor of the numerator.

Part (b)

Use long division to show that a - 2 is a factor of the denominator.

Part (c)

Use your information from Parts (a) and (b) to simplify the quotient. You will need to exclude a value of a.



Show that 3x + 4 is a factor of the numerator and the denominator in the quotient below, then use this information to simplify the quotient.

 $\frac{6x^4 + 8x^3 + 3x^2 + 7x + 4}{3x^3 + 4x^2 - 3x - 4}$