ALGEBRA REVIEW
UNIT 6 --- Working with Powers

In the previous units, we have worked with expressions raised to powers, but only to a limited extent. In this unit, we will take the time to introduce powers carefully. Like so many other concepts in algebra, the notion of raising an expression to a power develops from our intuition about positive integers.

If A is an expression, and n is a positive integer, then we understand A^n to represent the product of A with itself n times.

- In symbols, $A^n = \underbrace{A \cdot A \cdot A \cdot \cdots \cdot A}_{n \textit{ times}}$

When we multiply expressions that are raised to positive integer powers, certain patterns emerge.

Consider the product $(3a+5)^3 \cdot (3a+5)^2$. Based on our understanding of expressions raised to powers, we know

$$(3a+5)^3 \cdot (3a+5)^2 = \overbrace{(3a+5)\cdot(3a+5)\cdot(3a+5)}^{(3a+5)^3} \cdot \overbrace{(3a+5)\cdot(3a+5)}^{(3a+5)^2}$$

$$= (3a+5)^5$$

$$= (3a+5)^{3+2}$$

This pattern will hold whenever we take an expression raised to two positive integer powers and multiply --- the end result will be that expression raised to the *sum* of the powers. This pattern is the most important property of expressions raised to positive integer powers. Everything else we understand about powers is based upon it.

Additive Rule for Powers

(ARP) If A is any expression and m, n are positive integers, then

$$A^m \cdot A^n = A^{m+n}$$

Example 1. Rewrite $\frac{t^3}{x^4} - \frac{x^5}{yt^3}$ as a single quotient. Use ARP to simplify.

The common denominator will be yx^4t^3, so we need to adjust each term so that they all have this denominator.

$$\frac{t^3}{x^4} - \frac{x^5}{yt^3} = \frac{t^3}{x^4} \cdot 1 - \frac{x^5}{yt^3} \cdot 1$$

$$= \frac{t^3}{x^4} \cdot \left(\frac{yt^3}{yt^3}\right) - \frac{x^5}{yt^3} \cdot \left(\frac{x^4}{x^4}\right)$$ *(Apply RR1.)*

$$= \frac{t^3 \cdot t^3 y}{x^4 \cdot t^3 y} - \frac{x^5 \cdot x^4}{yt^3 \cdot x^4}$$ *(Apply MR2.)*

$$= \frac{t^3 \cdot t^3 y - x^5 \cdot x^4}{yx^4t^3}$$ *(Apply AR.)*

$$= \frac{t^6 y - x^9}{yx^4t^3}$$ *(Apply ARP.)*

Exercise 1 Follow Example 1 and rewrite $\frac{t}{p^2} - xp^4$ as a single quotient. Use ARP to simplify.

Exercise 2 Follow Example 1 and rewrite $\frac{h^3}{3c} + \frac{c^2}{h^4}$ as a single quotient. Use ARP to simplify.

Exercise 3 Rewrite $\frac{x^2}{(r-1)u^3} + \frac{(r-1)}{u^2} - 4u$ as a single quotient. Use ARP to simplify.

EXAMPLE 2. The numerator and denominator of the quotient below contain the same common factor. Simplify the expression by eliminating this common factor using ARP and Reciprocal Rule 1.

$$\frac{z^5-4xtz^3}{z^3+3yz^4}$$

Notice that each term in the numerator and denominator have a power of z as a factor. Furthermore, z^3 is the lowest power of z appearing in any of the terms. We can run ARP "backwards" to factor this power out of the numerator and denominator.

$$\frac{z^5-4xtz^3}{z^3+3yz^4} = \frac{z^{2+3} - 4xtz^3}{z^3 + 3y\,z^{1+3}}$$ *(Rewrite each power of z as a sum involving 3.)*

$$= \frac{z^2 \cdot z^3 - 4xt \cdot z^3}{z^3 + 3yz^1 \cdot z^3}$$ *(Apply ARP backward to write each power as a product containing z^3.)*

EXAMPLE 2 Continued

$$= \frac{(z^2 - 4xt) \cdot z^3}{(1+3yz) \cdot z^3}$$

(Factor z^3 out of numerator and denominator.)

$$= \left(\frac{z^2 - 4xt}{1+3yz}\right) \cdot \frac{z^3}{z^3}$$

(Apply MR2.)

$$= \left(\frac{z^2 - 4xt}{1+3yz}\right) \cdot 1 \quad (z \neq 0)$$

(Apply RR1 and exclude $z = 0$.)

$$= \frac{z^2 - 4xt}{1+3yz} \quad (z \neq 0)$$

Exercise 4 Follow the steps in Example 2 to simplify the quotient $\frac{rv^4 + yfv^6}{v^5}$.

Exercise 5 Follow the steps in Example 2 to simplify the quotient $\frac{4y^7 - y^3 + 6y}{y^5 - 2y}$.

Exercise 6 Follow the steps in Example 2 to simplify the quotient $\frac{3x^2 - tx^4}{x^3 - ytx^2 + 5x^4}$.

Exercise 7

Explain why the following equation is NOT valid.

$$\frac{a^3+3a^2}{a^3-a^2-y} \stackrel{?}{=} \frac{a+3}{a-1-y}$$

Exercise 8

Is the following equation valid? Justify your answer.

$$\frac{2x-x^2}{2x+1} \stackrel{?}{=} -x^2$$

Exercise 9

Simplify the quotient $\frac{(w-2)^2-4(w-2)^3}{(w-2)^3+5r^4(w-2)^6}$. Exclude values of variables as needed.

Exercise 10

The numerator and denominator of the expression below both have t^2-1 as a common factor. Use this fact to simplify the quotient. (You will need to use long division on the denominator.)

$$\frac{\left(t^2-1\right)^3-3\left(t^2-1\right)}{3t^4+2t^3-8t^2-2t+5}$$