ALGEBRA REVIEW
UNIT 8 --- Working with Rational Powers

We now have an understanding of what it means to take an expression and raise it to an integer power. It is also possible to develop a meaningful definition for *rational* powers of expressions, at least in some circumstances.

Before we do so, however, we need to introduce another rule for powers. This rule is really just a consequence of ARP, but it is almost as important in its own right.

Multiplicative Rule for Powers

MRP Let A be an expression, and suppose that $m. n$ are integers. It will always be true that

$$(A^m)^n \;=\; A^{m \cdot n}$$

as long as the power $(A^m)^n$ is defined.

To see why MRP is true for integer powers, consider the power $([2x-1]^{-2})^3$. We know

$$
\begin{aligned}
([2x-1]^{-2})^3 &= \left(\frac{1}{[2x-1]^2}\right)^3 \\
&= \frac{1}{[2x-1]^2} \cdot \frac{1}{[2x-1]^2} \cdot \frac{1}{[2x-1]^2} \\
&= \frac{1}{(2x-1)(2x-1)} \cdot \frac{1}{(2x-1)(2x-1)} \cdot \frac{1}{(2x-1)(2x-1)} \\
&= \frac{1}{(2x-1)\cdot(2x-1)\cdot(2x-1)\cdot(2x-1)\cdot(2x-1)\cdot(2x-1)} && \textit{(Apply MR2.)} \\
&= \frac{1}{(2x-1)^6} && \textit{(Apply ARP.)} \\
&= (2x-1)^{-6} \\
&= (2x-1)^{-2\cdot 3}
\end{aligned}
$$

Exercise 1

Simplify the expression $([3t-5]^3)^{-4}$ using MRP.

Exercise 2

Simplify the expression $([p^2-4t+9]^{-2})^{-7}$ using MRP.

Exercise 3

Explain why the equation $y^4 \cdot y^2 = y^8$ is not valid. What is the correct way to rewrite the left-hand side?

Exercise 4

Explain why the equation $(y^4)^4 = y^8$ is not valid. What is the correct way to rewrite the left-hand side?

Exercise 5

Simplify the quotient below as much as possible using MRP, ARP, and eliminating common factors from the numerator and denominator.

$$\frac{6ty^{-3}(x^2)^{-2}}{4t^{-1}}$$

Exercise 6

Simplify the quotient below as much as possible using MRP, ARP, and eliminating common factors from the numerator and denominator.

$$\frac{(m^3)^4\, m^{-3}}{y^{-2}\, m^5\, (x^3)^{-1}}$$

Let A and B be expressions, and suppose n is a positive integer. We say that B is an ***n th root*** of A provided

$$B^n = A$$

We sometimes use the special symbol $\sqrt[n]{A}$ to denote the expression B.

- The third root of 8 is the number 2, because $2^3 = 8$. In symbols, we write $\sqrt[3]{8} = 2$.
- The fifth root of −243 is the number −3 because $(-3)^5 = -243$. In symbols, we write $\sqrt[5]{-243} = -3$.
- The seventh root of 27 is *approximately* the number 1.6013 because $1.6013^7 = 26.99659091$. In symbols, we write $\sqrt[7]{27} \approx 1.6013$.

All of the examples above are *odd* roots. We started out with odd roots for examples because there are some special issues with *even* roots.

There is no fourth root of the number -16, because when we raise any nonzero number to the fourth power, the result is always positive.

When we raise a nonzero number to an *even* power, the result is always positive. Therefore, ***even roots of negative numbers do not exist***.

There are *two* fourth roots of the number 16. Since $2^4 = 16$ AND $(-2)^4 = 16$, both of these numbers are fourth roots of 16.

A positive number will always have two even roots, one that is positive and one that is negative. In this case, ***the nth root symbol always represents the positive even root.***

For example, we understand the symbol $\sqrt[2]{8}$ to represent the positive second root of 8 (which is approximately 2.828).

The second roots of a positive number are often called the *square* roots of that number, since raising a number to the second power is called "squaring."

It is common to see $\sqrt{A}$ used to represent the second root of an expression A.

Exercise 7 Why is it incorrect to write $\sqrt[4]{81} = -3$?

Exercise 8 Why is it incorrect to write $\sqrt{-4}$?

Exercise 9 If we simplify the expression $\sqrt{3^2}$, what would it be equal to?

Exercise 10 Is the expression $\sqrt[6]{(-4)^6}$ equal to 4, or is it equal to -4 ? Explain your answer.

EXAMPLE 1. For what values of p will the expression $\sqrt[4]{2-5p}$ be defined?

The even root of an expression is defined only when the expression is nonnegative; therefore, we are interested in solving the inequality $2-5p \geq 0$.

$$2-5p \geq 0 \Rightarrow (2-5p)+5p \geq 0+5p \quad \textit{(Add 5p to both sides.)}$$

$$\Rightarrow 2+(5-5)p \geq 5p \quad \textit{(Combine like terms.)}$$

$$\Rightarrow 2 \geq 5p \quad \textit{(Simplify.)}$$

$$\Rightarrow \frac{2}{5} \geq \frac{5p}{5} \quad \textit{(Divide both sides by 5.)}$$

$$\Rightarrow \frac{2}{5} \geq p \quad \textit{(Simplify.)}$$

The expression is defined for all values of p that are less than or equal to 2/5.

Solving a linear inequality follows the same steps used in solving a linear equality. However, before dividing both sides of an inequality by a number, it is wise to arrange the inequality so that the number is positive (like we did in Example 1).

When you divide both sides of an inequality by a negative number, you must reverse the inequality.

EXAMPLE 1b. Solving the inequality $2-5p \geq 0$ --- an alternative approach.

$$2-5p \;\geq\; 0 \quad \Rightarrow \quad (2-5p) - 2 \;\geq\; 0-2 \qquad \textit{(Add 2 to both sides.)}$$

$$\Rightarrow \quad -5p + 0 \;\geq\; -2 \qquad \textit{(Combine like terms.)}$$

$$\Rightarrow \quad -5p \;\geq\; -2 \qquad \textit{(Simplify.)}$$

$$\Rightarrow \quad \frac{-5p}{-5} \;\leq\; \frac{-2}{-5} \qquad \textit{(Divide both sides by 5 and reverse inequality.)}$$

$$\Rightarrow \quad p \;\leq\; \frac{2}{5} \qquad \textit{(Simplify.)}$$

Exercise 11 For what values of x will the expression $\sqrt{4x-9}$ be defined?

Exercise 12 For what values of u will the expression $\sqrt[6]{12-7u}$ be defined?

Exercise 13 For what values of t will the expression $\frac{1}{\sqrt[4]{3t+8}}$ be defined?

Exercise 14 Explain why the expression $\sqrt[3]{7s+6}$ is defined for all values of s.

We are now ready to consider *rational* powers of an expression. A ***rational number*** is the quotient of two integers; and since we have developed a meaning for any integer power Of an expression, it is natural to wonder if we can give some kind of meaning to rational powers of expressions.

We will define rational powers of an expression in such a way that MRP will still be valid.

Suppose A is an expression for which $\sqrt[n]{A}$ is defined. We know that

$$\left(\sqrt[n]{A}\right)^n = A = A^1$$

We want to define the rational power $A^{1/n}$ in such a way that MRP will be valid whenever we take the expression $A^{1/n}$ and raise it to a power. In particular, we want

$$\left(A^{1/n}\right)^n = A^{n \cdot 1/n} = A^1$$

This tells us that we should **define** the rational power $A^{1/n}$ to be the nth root of A .

Let A be any expression such that $\sqrt[n]{A}$ is defined. In this case, we **define** the power $A^{1/n}$ by

$$A^{1/n} = \sqrt[n]{A}$$

- We define $A^{1/n}$ in this way so that MRP will also be valid for these powers.

This definition for rational powers of an expression is also consistent with ARP.

EXAMPLE 2. Simplify the quotient $\frac{4t\,(2x-1)^{-2}\;t^{1/4}}{(2x-1)^{1/3}}$ as much as possible.

$$\frac{4t\,(2x-1)^{-2}\;t^{1/4}}{(2x-1)^{1/3}} = 4 \cdot t^{1+1/4} \cdot (2x-1)^{-2-1/3}$$ *(Combine powers using ARP.)*

$$= 4 \cdot t^{5/4} \cdot (2x-1)^{-7/3}$$ *(Add the powers together.)*

$$= \frac{4t^{5/4}}{(2x-1)^{7/3}}$$ *(Rewrite so that all powers are positive.)*

The last example leaves us with a question --- What do the expressions $t^{5/4}$ and $(2x-1)^{7/3}$ actually mean?

Since we want MRP to be valid for rational powers, we want

$$\left(t^{5/4}\right)^4 = t^{4\cdot 5/4} = t^5 \qquad \left[(2x-1)^{7/3}\right]^3 = (2x-1)^{3\cdot 7/3} = (2x-1)^7$$

This tells us that we want $t^{5/4}$ to be the fourth root of t^5 , and we want $(2x-1)^{7/3}$ to be the third root of $(2x-1)^7$.

Exercise 15 Simplify the quotient $\frac{14x^2}{yx^{1/3}}$ as much as possible.

Exercise 16 Simplify the quotient $\frac{3ru^{1/2}}{4r^{-2}u^3}$ as much as possible.

Exercise 17 Simplify the quotient $\frac{m\,(r-3)^{2/3}\,x^{-1}}{6x^{1/2}\,m^{3/4}\,(r-3)^{1/3}}$ as much as possible.

Exercise 18 Use specific values for t and u to show that the following equation is not valid.

$$(t^2+u^2)^{1/2} \overset{?}{=} t+u$$

Exercise 19 Is the equation $\sqrt{a^2+b^2} = a+b$ always valid? Justify your answer.

Exercise 20 Is the equation $r^{3/2} = r\sqrt{r}$ always valid? Justify your answer.

Exercise 21 If we simplify the expression $[(-3)^4]^{1/4}$ what number would we obtain?