Up to now, we have been concerned with constructing the derivative function for a known function f. In the sciences, it is often more important to *go the other way* --- given the derivative function for some unknown function f, construct the function itself. As we shall see, this is a trickier process.

Problem 1. Show that the derivative function for $y = F(x) = x\ln(x) - x$ is the function defined by $r = F'(x) = \ln(x)$.

Problem 2. Show that the derivative function for $y = F(x) = \frac{\sin(x) - 4\cos(x)}{\cos(x)}$ is the function defined by $r = F'(x) = \sec^2(x)$

Problem 3. The function $y = f(x) = \ln(x)$ is only defined when $x > 0$.

Part (a). Explain why the function $y = F(x) = \ln|x|$ is defined for all nonzero values of x.

Part (b). Explain why, when $x < 0$, we have $\ln|x| = \ln(-x)$.

Part (c). Use the Chain Rule to determine the formula for $\frac{d}{dx}[\ln(-x)]$.

Problem 4. Explain why $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$ for all nonzero values of x.

> ***Antiderivative***
>
> Let $r = f(x)$ be a function. We say that a function $y = F(x)$ is an *antiderivative* for the function f provided
> $$F'(x) = f(x)$$

In Problems 1 - 4, you showed that

$y = F(x) = x\ln(x) - x$ is one antiderivative for the function $r = f(x) = \ln(x)$

$y = F(x) = \frac{\sin(x) - 4\cos(x)}{\cos(x)}$ is one antiderivative for the function $r = f(x) = \sec^2(x)$

$y = F(x) = \ln|x|$ is one antiderivative for the function $r = f(x) = \frac{1}{x}$

Now, according to the special derivative formulas, we know that

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

Consequently, we actually know of *two* antiderivatives for the function $r = f(x) = \sec^2(x)$.

Problem 4. Suppose you know that $y = F(x)$ is one antiderivative for a function $r = f(x)$. If C represents any constant, explain why the function

$$y = F_C(x) = F(x) + C$$

is also an antiderivative for the function f. (What happens if you differentiate the function F_C?)

The two antiderivatives we know of for the function $r = f(x) = \sec^2(x)$ may look very different. However, observe that

$$\frac{\sin(x) - 4\cos(x)}{\cos(x)} = \frac{\sin(x)}{\cos(x)} - \frac{4\cos(x)}{\cos(x)} = \tan(x) - 4$$

Therefore, the two antiderivatives for the function $r = f(x) = \sec^2(x)$ really only differ by a constant.

The Antiderivative Family for a Function

Let $r = f(x)$ be a function. If $y = F(x)$ and $y = G(x)$ are antiderivatives for the function f, then

$$G(x) - F(x) = C$$

for some constant C. It is customary to use the symbol

$$\int f(x)dx$$

to represent the family of *all* antiderivatives for the function f.

Based on Problem 1, we now know

$$\int \ln(x)\, dx = \{G(x) = x\ln(x) - x + C : C \text{ is any constant}\}$$

This notation is read "*The antiderivative family for the function $r = f(x) = \ln(x)$ is the set of all functions $G(x) = x\ln(x) - x + C$ such that C is any constant.*"

It is customary (but not really correct) to abbreviate the notation above to

$$\int \ln(x)\, dx = x\ln(x) - x + C$$

The abbreviated notation is still read the same way.

Problem 5. Use the customary notation to write the antiderivative families for the functions $y = f(x) = \sec^2(x)$ and $= f(x) = \frac{1}{x}$.

Sometimes, we can apply the general derivative rules *in reverse* to determine antiderivative families. For example, suppose we want to know the antiderivative family for the function $r = f(x) = x^2$.

$$\frac{d}{dx}[x^3] = 3x^2 \quad \Rightarrow \quad \left(\frac{1}{3}\right)\frac{d}{dx}[x^3] = x^2$$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$$ ***(Apply the Constant Multiple Rule in reverse.)***

$$\Rightarrow \quad \int x^2\, dx = \frac{1}{3}x^3 + C$$

Problem 6. Use the fact that $\frac{d}{dx}[\cos(x)] = (-1)\sin(x)$ to find the antiderivative family for the function $r = f(x) = \sin(x)$.

Problem 7. Use the fact that $\frac{d}{dx}[x^{n+1}] = (n+1)x^n$ to find the antiderivative family for the function $r = f(x) = x^n$ (as long as $n \neq -1$).

Special Antiderivative Formulas

1. Since $\frac{d}{dx}[Kx] = K$ for any constant K, we know $\int K\,dx = Kx + C$.
2. Since $\frac{d}{dx}[\sin(x)] = \cos(x)$, we know $\int \cos(x)\,dx = \sin(x) + C$.
3. Since $\frac{d}{dx}[\cos(x)] = -\sin(x)$, we know $\int \sin(x)\,dx = -\cos(x) + C$.
4. Since $\frac{d}{dx}[\tan(\mathrm{x})] = \sec^2(x)$, we know $\int \sec^2(x)\,dx = \tan(x) + C$.
5. Since $\frac{d}{dx}[x\ln(x) - x] = \ln(x)$, we know $\int \ln(x)\,dx = x\ln(x) - x + C$.
6. Since $\frac{d}{dx}[\ln|x|] = x^{-1}$, we know $\int x^{-1}\,dx = \ln|x| + C$.
7. Since $\frac{d}{dx}[x^{n+1}] = (\mathrm{n}+1)x^n$, when $n \neq -1$, we know $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$.
8. Since $\frac{d}{dx}[a^x] = a^x\ln(a)$, we know $\int a^x\,dx = \frac{1}{\ln(a)} \cdot a^x + C$.
9. Since $\frac{d}{dx}[\mathrm{Arcsin}(x)] = \frac{1}{\sqrt{1-x^2}}$, we know $\int \frac{1}{\sqrt{1-x^2}}\,dx = \mathrm{Arcsin}(x) + C$.
10. Since $\frac{d}{dx}[\mathrm{Arctan}(x)] = \frac{1}{1+x^2}$, we know $\int \frac{1}{1+x^2}\,dx = \mathrm{Arctan}(x) + C$.

Example 1. What is the antiderivative family for the function $r = f(x) = 2\cos(x) + 5x^4$?

Solution. We can approach this problem by applying the general derivative rules in reverse. Observe

$$f(x) = 2\cos(x) - x^4 \quad \Rightarrow \quad f(x) = 2\frac{d}{dx}[\sin(x)] + \frac{d}{dx}[x^5]$$ *(Apply Specific Derivative Formulas in reverse)*

$$\Rightarrow \quad f(x) = \frac{d}{dx}[2\sin(x)] + \frac{d}{dx}[x^5]$$ *(Apply Constant Multiple Rule in reverse)*

$$\Rightarrow \quad f(x) = \frac{d}{dx}[2\sin(x) + x^5]$$ *(Apply Sum Formula in reverse.)*

$$\Rightarrow \quad \int [2\cos(x) - x^4]dx = 2\sin(x) + x^5 + C$$

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We can summarize the results of reversing the general derivative rules using the following general *antiderivative* rules.

General Antiderivative Rules

Suppose that $r = f(x)$ and $r = g(x)$ are functions, and suppose that K is a fixed constant.

- Anti-Constant-Multiple Rule: $\int Kf(x)\,dx = K\int f(x)\,dx$
- Anti-Sum Rule: $\int [f(x) + g(x)]dx = \int f(x)\,dx + \int g(x)\,dx$

Problem 8. Use the general antiderivative rules and specific antiderivative formulas to evaluate

$$\int \left[3\sin(x) + \frac{2}{x} - 5\right] dx$$

Problem 9. Determine the antiderivative family for the function $f(x) = 3e^x - 4\ln(x)$.

Problem 10. Evaluate $\int \left[2b^{-2} + 5b^{1/2} - 3\right] db$.

Problem 11. Find the antiderivative family for the function $m = f(n) = 3n^{-3/4} + 2\sec^2(n)$.

Problem12. Evaluate $\int \left[\frac{2}{\sqrt{1-t^2}} - \pi \ln(t)\right] dt$.

The Anti-Chain Rule

Let $r = f(x)$ be a function. If it is possible to find functions $r = g(u)$ and $u = h(x)$ so that

$$f(x) = g(u) \cdot \frac{du}{dx}$$

then the following equation is true:

$$\int f(x)\, dx = \int \left[g(u) \cdot \frac{du}{dx}\right] dx = \int g(u) du$$

Example 2. Use the Anti-chain Rule to find the antiderivative family for $r = f(x) = 2x\sin(x^2)$.

Solution. If we let $u = h(x) = x^2$, then $\frac{du}{dx} = 2x$. Consequently, we know

$$\int 2x\sin(x^2)\, dx = \int \sin(u) \cdot [2x]\, dx = \int \sin(u) \cdot \left[\frac{du}{dx}\right] dx = \int \sin(u)\, du$$

We have now recast the antiderivative problem so that it exactly matches one of the special antiderivative formulas. Therefore, we know

$$\int 2x\sin(x^2)\, dx = \int \sin(u)\, du = -\cos(u)|_{u=x^2} + C = -\cos(x^2) + C$$

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Problem 13. Use the Anti-chain Rule to find the antiderivative family for $f(x) = 4x^3 \cos(x^4)$.

Problem 14. Use the Anti-chain Rule to evaluate $\int \frac{1}{x}(2 + \ln(x))^3 \, dx$.

Example 3. Use the Anti-chain Rule to find the antiderivative family for $r = f(x) = x^2\sqrt{1 + 4x^3}$.

Solution. In this case, let $u = h(x) = 1 + 4x^3$. This tells us

$$\frac{du}{dx} = 12x^2 \qquad \text{so that} \qquad \frac{1}{12} \cdot \frac{du}{dx} = x^2$$

Consequently, we know

$$\begin{aligned}
\int x^2\sqrt{1 + 4x^3}\, dx &= \int \sqrt{u} \cdot [x^2] dx \\
&= \int \sqrt{u} \cdot \left[\frac{1}{12} \cdot \frac{du}{dx}\right] dx \\
&= \frac{1}{12} \int u^{1/2} \cdot \left[\frac{du}{dx}\right] dx \\
&= \frac{1}{12} \int u^{1/2}\, du \\
&= \left(\frac{1}{12}\right)\left(\frac{1}{1/2 + 1}\right) u^{1/2+1}\Big|_{u=1+4x^3} + C \\
&= \frac{1}{8}(1 + 4x^3)^{3/2} + C
\end{aligned}$$

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Problem 15. Evaluate $\int x\sin(x^2)dx$.

Problem 16. Find the antiderivative family for the function $f(x) = \frac{x^2}{2+x^3}$.

Hint: Note that $f(x) = x^2(2 + x^3)^{-1}$.

Homework

1. Show by differentiation that the function $y = F(x) = \frac{e^x}{2}[\sin(x) - \cos(x)]$ is one antiderivative for the function $r = f(x) = e^x \sin(x)$.

2. Show by differentiation that $y = F(x) = \ln(1 + x^2) + \text{Arctan}(x)$ is one antiderivative for the function
$$r = f(x) = \frac{x+1}{1+x^2}$$

Evaluate each of the following.

(3) $\int [2e^x - 3\cos(x)]dx$

(4) $\int [3\ln(t) + 5t^{-2}]\, dt$

(5) $\int \left[2v^{1/3} - \sin(v)\right]dv$

(6) $\int \left[\frac{4}{x} + \frac{3}{\sqrt{1-x^2}} + \frac{5}{2}\right] dx$

(7) $\int t^3 \ln(t^4) dt$

(8) $\int \sin(v) \sec^2(\cos(v)) dv$

(9) $\int (x-1)(x^2 - 2x)^3 dx$

(10) $\int \frac{t}{\sqrt{1-t^2}} dt$

(11) $\int \frac{2v^2+3}{2v^3+9v} dv$

(12) $\int \frac{\sin(x)}{\cos(x)} dx$

(13) $\int \left[t^{-3/4} + t\sin(t^2)\right] dt$

Answers.

Differentiate the functions F given in Problems 1 and 2 and show that $F' = f$ in both cases.

(3) $\int [2e^x - 3\cos(x)]dx = 2e^x - 3\sin(x) + C$

(4) $\int [3\ln(t) + 5t^{-2}]\, dt = 3(t\ln(t) - t) - 5t^{-1} + C$

(5) $\int \left[2v^{1/3} - \sin(v)\right]dv = \frac{3}{2}v^{4/3} + \cos(v) + C$

(6) $\int \left[\frac{4}{x} + \frac{3}{\sqrt{1-x^2}} + \frac{5}{2}\right] dx = 4\ln|x| + 3\text{Arcsin}(x) + \frac{5}{2}x + C$

(7) $\int t^3 \ln(t^4) dt = \frac{1}{4}[t^4 \ln(t^4) - t^4] + C$

(8) $\int \sin(v) \sec^2(\cos(v)) dv = -\tan(\cos(v)) + C$

(9) $\int (x-1)(x^2 - 2x)^3 dx = \frac{1}{8}(x^2 - 2x)^4 + C$

(10) $\int \frac{t}{\sqrt{1-t^2}}\,dt = -\frac{1}{2}\sqrt{1-t^2} + C$

(11) $\int \frac{2v^2+3}{2v^3+9v}\,dv = \frac{1}{3}\ln|2v^3 + 9v| + C$

(12) $\int \frac{\sin(x)}{\cos(x)}\,dx = -\ln|\cos(x)| + C$

(13) $\int \left[t^{-3/4} + t\sin(t^2)\right] dt = 4t^{1/4} - \frac{1}{2}\cos(t^2) + C$