REVERSING DIFFERENTIATION

A *differential equation* is an equation involving a function and/or one or more of its derivatives. The solution to a differential equation will be a function, or a set of functions. For example, the differential equation

$$y'' = x$$

has the function $f(x) = \dfrac{x^3}{6} + 5$ as one solution. At this stage, there is only one way to see this — we need to show that the second derivative of f is equal to x. Observe

- $f'(x) = \frac{d}{dx}\left[\dfrac{x^3}{6} + 5\right] = \left(\dfrac{1}{6}\right)\left(3x^2\right) + 0 = \dfrac{x^2}{2}$
- $f''(x) = \frac{d}{dx}\left[\dfrac{x^2}{2}\right] = \left(\dfrac{1}{2}\right)(2x) = x$

Since the proposed formula for f makes the differential equation true, we know that this formula is one solution to the differential equation. There might be more.

Problem 1. Show that the function $f(x) = \dfrac{x^3}{6} + K$ is a solution to the differential equation $y'' = x$ for *any* constant K.

Example 1 *Show by direct computation that the function $f(x) = x^2 + e^x$ is a solution to the differential equation $y' - y'' = 2(x-1)$.*

Solution. The differential equation tells us we are looking for functions f with the property that their first derivative minus their second derivative is equal to $2(x-1)$. We need to show that the proposed function f has this property. First, observe that

- $f'(x) = \frac{d}{dx}\left[x^2 + e^x\right] = 2x + e^x$
- $f''(x) = \frac{d}{dx}\left[2x + e^x\right] = 2 + e^x$

Now, we construct the difference between these derivatives. Observe

$$\begin{aligned} f'(x) - f''(x) &= (2x + e^x) - (2 + e^x) \\ &= 2x + e^x - 2 - e^x \\ &= 2x - 2 \\ &= 2(x-1) \end{aligned}$$

Therefore, the function f satisfies the differential equation.

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Problem 2. Show by direct computation that the function $f(x) = \cos(x)$ is a solution to the differential equation $2y'' - y = -3\cos(x)$.

There are entire courses devoted to techniques for solving differential equations, and the details of these techniques depend on the type of differential equation being considered. In this discussion, we will restrict our attention to the most basic class of differential equation.

- An nth-order *basic* differential equation has the form $y^{(n)} = g(x)$, where g is a specified function of x, and $y^{(n)}$ represents the nth derivative of y with respect to x. The goal is always to find a function $y = f(x)$ that makes the differential equation true.

There is a systematic approach to determining the solutions to an nth-order basic differential equation, and it begins with a definition.

ANTIDERIVATIVE FAMILIES

- We say that a function G is an *antiderivative* for a given function g provided G solves the first-order basic differential equation $y' = g(x)$. The set of all solutions to this differential equation is called the *antiderivative family* for the function g.

In other words, G is an antiderivative for g provided g is the derivative function for G. Our special derivative formulas give us solutions to a host of first-order basic differential equations.

Example 2 *What is an antiderivative for the function $g(x) = x^4$?*

We want to find a function G whose derivative is $g(x) = x^4$. Now, we know from the power rule that

$$\begin{aligned} \frac{d}{dx}\left[x^5\right] = 5x^4 \quad &\Longrightarrow \quad \left(\frac{1}{5}\right)\frac{d}{dx}\left[x^5\right] = x^4 && \text{(Solve the equation for } x^4\text{.)} \\ &\Longrightarrow \quad \frac{d}{dx}\left[\frac{x^5}{5}\right] = x^4 && \text{(Apply the Constant Multiple Rule \textit{in reverse}.)} \end{aligned}$$

Therefore, we know that $G(x) = \dfrac{x^5}{5}$ is one antiderivative for g.

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Example 3 *What is an antiderivative for the function $g(x) = \sin(x)$?*

We want to find a function G whose derivative is $g(x) = \sin(x)$. Now, we know from the specific derivative formulas list that

$$\begin{aligned} \frac{d}{dx}[\cos(x)] = (-1)\sin(x) \quad &\Longrightarrow \quad (-1)\frac{d}{dx}[\cos(x)] = \sin(x) \qquad \text{(Solve the equation for } \sin(x).) \\ &\Longrightarrow \quad \frac{d}{dx}[(-1)\cos(x)] = \sin(x) \qquad \text{(Apply the Constant Multiple Rule \textit{in reverse}.)} \end{aligned}$$

Therefore, we know that $G(x) = -\cos(x)$ is one antiderivative for g.

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Problem 3. Use the special derivative formulas to help you find an antiderivative for each of the following functions.

(a) $f(x) = x^3$ (b) $g(x) = \cos(x)$ (c) $h(x) = e^x$

There is more to the story in Example 1, however. Notice that if C is any fixed real number, then

$$\frac{d}{dx}\left[\frac{x^5}{5} + C\right] = \frac{d}{dx}\left[\frac{x^5}{5}\right] + \frac{d}{dx}[C] = x^4 + 0 = x^4$$

Therefore, we can add *any fixed constant* to an antiderivative for a function and create another anderivative for that same function. The antiderivative family for a given function is always an infinite set.

Example 4 *Show that $F(\theta) = \tan(\theta)$ and $G(\theta) = \dfrac{\sin(\theta) - 3\cos(\theta)}{\cos(\theta)}$ are both antiderivatives for $g(\theta) = \sec^2(\theta)$.*

Solution. First, the special derivative formulas tell us that, for any constant K, we have

$$\frac{d}{d\theta}[\tan(\theta) + K] = \sec^2(\theta)$$

The functions F and G do not look very much alike. However, looks can be deceiving. Observe that

$$\begin{aligned} G(\theta) &= \frac{\sin(\theta) - 3\cos(\theta)}{\cos(\theta)} \\ &= \frac{\sin(\theta)}{\cos(\theta)} - \frac{3\cos(\theta)}{\cos(\theta)} \\ &= \tan(\theta) - 3 \end{aligned}$$

Consequently, the functions F and G only differ by a constant and therefore must have the same derivative.

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- Any two antiderivatives for a function f will differ by at most a constant. In other words, the antiderivatives for a function f are all vertical translations of each other.

If you can determine one antiderivative for a function f, then you create every antiderivative for f simply by adding a fixed constant to the antiderivative that you determined. The antiderivative corresponding to the constant $C = 0$ is called the *fundamental* antiderivative for f.

- We use a special notation to indicate that we are constructing the antiderivative family for a function f. We let

$$\int f(x)dx$$

represent the antiderivative family for the function f. This is only a notational symbol; we will motivate its use in the next unit. For example,

$$\int x^4 dx = \frac{x^5}{5} + C \qquad \int \sec^2(\theta)d\theta = \tan(\theta) + K$$

where C and K represent all possible real numbers.

SPECIFIC ANTIDERIVATIVE FORMULAS

- $\frac{d}{dx}[\sin(x)] = \cos(x)$ implies $\int \cos(x)dx = \sin(x) + C$
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$ implies $\int \sin(x)dx = -\cos(x) + C$
- $\frac{d}{dx}[\tan(x)] = \sec^2(x)$ implies $\int \sec^2 dx = \tan(x) + C$
- $\frac{d}{dx}[e^x] = e^x$ implies $\int e^x dx = e^x + C$
- $\frac{d}{dx}[x\ln(x) - x] = \ln(x)$ implies $\int \ln(x)dx = x\ln(x) - x + C$
- $\frac{d}{dx}[Kx] = K$ implies $\int K dx = Kx + C$ for any constant K
- $\frac{d}{dx}[x^r] = rx^{r-1}$ implies $\int x^r dx = \frac{x^{r+1}}{r+1}x^{r+1} + C$ so long as $r \neq -1$

Example 5 *Show that the fundamental antiderivative for $f(x) = x^{-1}$ is $F(x) = \ln|x|$.*

Solution. We will show that this is the case by differentiating the function F. We will need the chain rule. Let $u(x) = |x|$ and let $g(u) = \ln(u)$. Now, we know that

$$\frac{dg}{du} = \frac{1}{u} = \frac{1}{|x|}$$

Computing the derivative of u requires the definition of the absolute value function. Recall

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

With this in mind, we can see that

$$\frac{d}{dx}\left[|x|\right] = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

The derivative of the absolute value function is not defined when $x = 0$. However, this is not a problem, since $f(x) = \ln|x|$ is also not defined when $x = 0$. Now, we see

$$\begin{aligned} \frac{d}{dx}\left[\ln|x|\right] &= \frac{d}{du}\left[\ln(u)\right]\frac{d}{dx}\left[|x|\right] \\ &= \begin{cases} \frac{1}{|x|} & \text{if} \quad x > 0 \\ -\frac{1}{|x|} & \text{if} \quad x < 0 \end{cases} \end{aligned}$$

However, since $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we see that

$$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}$$

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Example 6 *Construct the antiderivative family for the function* $f(x) = x^{-3} + 4x^{-1}$.

Solution. We will construct this antiderivative family in steps. First, observe that the specific antiderivative formulas tell us

$$\int x^{-3}dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{x^{-2}}{2} + C \qquad \int x^{-1}dx = \ln|x| + K$$

where C and K can take on any real real number as their value.. Now, the constant multiple rule for differentiation also tells us

$$\frac{d}{dx}\left[4\ln|x|\right] = 4\frac{d}{dx}\left[\ln|x|\right] = 4x^{-1}$$

Consequently, the antiderivative family for $g(x) = 4x^{-1}$ is given by

$$\int 4x^{-1}dx = 4\int x^{-1}dx = 4\ln|x| + D$$

where D can take on any real real number as its value.. Finally, the sum rule and constant multiple rules for differentiation tells us

$$\frac{d}{dx}\left[-\frac{x^{-2}}{2} + 4\ln|x|\right] = \left(-\frac{1}{2}\right)\frac{d}{dx}\left[x^{-2}\right] + 4\frac{d}{dx}\left[\ln|x|\right] = x^{-3} + 4x^{-1}$$

Therefore, we know

$$\begin{aligned} \int \left(x^{-3} + 4x^{-1}\right) dx &= -\frac{x^{-2}}{2} + 4\ln|x| + E \\ &= \int x^{-3}dx + 4\int x^{-1}dx \end{aligned}$$

where E can take on any real real number as its value.

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The method we used for solving the initial value problem in Example 3 highlights some general rules for constructing antiderivative families:

Anti Constant Multiple Rule: If c is any constant, then $\int c \cdot f(x)dx = c\int f(x)dx$.

Anti Sum Rule: We have $\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$.

Problem 4. Use the general antiderivative rules and the specific antiderivative formulas to determine the antiderivative family for $f(x) = 2\ln(x) - \frac{3}{x}$.

Problem 5. Use the general antiderivative rules and the specific antiderivative formulas to determine the antiderivative family for $f(x) = \frac{2}{x^2} - 4\sin(x)$.

Problem 6. Compute $\int \left[r^6 + 5\cos(r)\right] dr$.

Problem 7. Compute $\int \left[9 - 4t^{-2} + 4\ln(t)\right] dt$.

HOMEWORK: Section 4.9 Page 355 Problems 1, 3, 7, 9, 10, 11, 13, 17

Antidifferentiation, in a sense, *reverses* the act of differentiating a function. What this tells us is, for any differentiable function f, we have

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x) \qquad \text{AND} \qquad \int \left[\frac{df}{dx}\right] dx = f(x) + C$$

Example 7 *What are the solutions to the basic differential equation $y'' = 3x + \cos(x)$?*

Solution. To begin, it is helpful to remember that the second derivative of a function f is really the derivative of the first derivative for f. Consequently, we can rewrite the differential equation as

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = 3x + \cos(x)$$

It is not necessary to do this, but it does help to clarify the method for obtaining the solution. Now, apply the act of antidifferentiation to both sides of this equation:

$$\begin{aligned}
\int \left(\frac{d}{dx}\left[\frac{dy}{dx}\right]\right) dx = \int (3x + \cos(x))\, dx \quad &\Longrightarrow \quad \frac{dy}{dx} + C = 3\int x dx + \int \cos(x) dx \\
&\Longrightarrow \quad \frac{dy}{dx} + C = \frac{3x^2}{2} + \sin(x) + K
\end{aligned}$$

where K and C represent any real numbers. Now, apply the act of antidifferentiation to both sides of this equation:

$$\begin{aligned}
\frac{dy}{dx} + C = \frac{3x^2}{2} + \sin(x) + K \quad &\Longrightarrow \quad \int \left[\frac{dy}{dx} + C\right] dx = \int \left[\frac{3x^2}{2} + \sin(x) + K\right] dx \\
&\Longrightarrow \quad \int \left[\frac{dy}{dx}\right] dx + \int C dx = \frac{3}{2}\int x^2 dx + \int \sin(x) dx + \int K dx \\
&\Longrightarrow \quad y + Cx + D = \left(\frac{3}{2}\right)\left(\frac{x^3}{3}\right) - \cos(x) + Kx + E
\end{aligned}$$

where D and E can take on any real real number as its value. Now, solving the last equation for y gives us

$$y = \frac{x^3}{2} - \cos(x) + (K - C)\, x + (E - D)$$

Since the constants $C, D, E,$ and K are all completely arbitrary real numbers, we usually just rename the differences in the above solution:

$$y = \frac{x^3}{2} - \cos(x) + Mx + N$$

where M and N can each independently take on any real number as their value.

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Problem 8. What are the solutions to the differential equation $y'' = x^{-1} + 4x^2 + 3$?

Example 8 *What is the particular antiderivative that satisfies $y' = \ln(x)$ if we also require that $y(1) = 10$?*

Solution. First, we determine the full antiderivative family. Observe

$$\int \left[\frac{dy}{dx}\right] dx = \int \ln(x)dx \Longrightarrow y = x\ln(x) - x + C$$

Notice that this time, we used only one arbitrary constant instead of two since we would have ended up combining them anyway. The extra condition that $y(1) = 10$ allows us to narrow down the antiderivative family to a single function by determining which value of the constant C produces the desired output. If we also require that $y(1) = 10$, this tells us

$$\begin{aligned} 10 = y(1) \quad &\Longrightarrow \quad 10 = (1)\ln(1) - 1 + C \\ &\Longrightarrow \quad 10 = (1)(0) - 1 + C \\ &\Longrightarrow \quad 11 = C \end{aligned}$$

The function $y = x\ln(x) - x + 11$ is the particular antiderivative that we seek.

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Example 8 is called an *initial value problem* (or IVP). An initial value problem is a differential equation coupled with specified outputs for the function and/or its derivatives.

Problem 9. Solve the initial value problem $y' = \sin(x)$ if we require $y(0) = -4$.

Example 9 *Solve the initial value problem $y'' = x$ where we require $y'(2) = 5$ and $y(3) = -10$.*

Solution. We "unwind" the second derivative in two steps. First, observe that

$$\begin{aligned} y'' = x \quad &\Longrightarrow \quad \int \left[\frac{d}{dx}\left[\frac{dy}{dx}\right]\right] dx = \int x dx \\ &\Longrightarrow \quad y' = \frac{dy}{dx} = \frac{x^2}{2} + C \end{aligned}$$

Since we have required $y'(2) = 5$, we know

$$\begin{aligned} 5 = y'(2) \quad &\Longrightarrow \quad 5 = \frac{2^2}{2} + C \\ &\Longrightarrow \quad 5 = 2 + C \\ &\Longrightarrow \quad 3 = C \end{aligned}$$

Therefore, we want to use the particular function $y' = \dfrac{x^2}{2} + 3$ in our next step.

We now want to solve the initial value problem $y' = \dfrac{x^2}{2} + 3$ with the requirement that $y(3) = -10$. Observe

To begin, we need the fundamental antiderivative family for G. We know

$$\begin{aligned} y' = \frac{x^2}{2} + 3 \quad &\Longrightarrow \quad \int \left[\frac{dy}{dx}\right] dx = \int \left(\frac{x^2}{2} + 3\right) dx \\ &\Longrightarrow \quad y = \left(\frac{1}{2}\right) \int x^2 dx + \int 3dx \\ &\Longrightarrow \quad y = \frac{x^3}{6} + 3x + D \end{aligned}$$

where D can take on any real real number as its value. Therefore, since we have required that $y(3) = -10$, we know

$$\begin{aligned} -10 = y(3) \quad &\Longrightarrow \quad -10 = \frac{3^3}{6} + 3(3) + D \\ &\Longrightarrow \quad -10 = \frac{9}{2} + 9 + D \\ &\Longrightarrow \quad -\frac{47}{2} = D \end{aligned}$$

The particular antiderivative we seek is $y = \dfrac{x^3}{6} + 3x - \dfrac{47}{2}$.

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Problem 10. Solve the initial value problem $y'' = 5$ if we require $y'(1) = 0$ and $y(2) = 3$.

HOMEWORK: Section 4.9 Page 356, Problems 25, 26, 27, 28, 31, 35, 38, 39, 41, 43

Notice that the anti-sum and anti-constant-multiple-rule mimic the sum and constant multiple rules for differentiation. There is also an anti-product rule, and an anti-chain rule. In this course, we will only concern ourselves with the anti-chain rule.

Anti Chain Rule: If it is possible to write a function $y = f(x)$ in the form $y = g(u) \cdot \frac{du}{dx}$, for some function g, then

$$\int f(x)dx = \int g(u)\left(\frac{du}{dx}\right)dx = \int g(u)du$$

In the anti-chain rule, the function u along with u' is called a *function-derivative pair.* The anti-chain rule can only be applied to functions in which we can find a function-derivative pair.

Example 10 *Use the anti-chain rule to determine the antiderivative family for* $f(x) = 2x\sin(x^2)$*.*

Solution. In this case, if we let $u = x^2$, then $u' = 2x$, and we have $f(x) = \sin(u) \cdot \frac{du}{dx}$. Therefore, the anti-chain rule tells us

$$\begin{aligned}
\int 2x\sin(x^2)dx &= \int \left[\sin(u) \cdot \frac{du}{dx}\right]dx \\
&= \int \sin(u)du \\
&= -\cos(u) + C \\
&= -\cos(x^2) + C
\end{aligned}$$

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Problem 11. Use the anti-chain rule to determine the antiderivative family for $f(x) = \cos(x)\sec^2(\sin(x))$.

Example 11 *Use the anti-chain rule to determine the antiderivative family for* $f(x) = x^2e^{x^3}$*.*

Solution. In this case, if we let $u = x^3$, then $\frac{du}{dx} = 3x^2$. This is *almost* a function derivative pair for f. Notice that

$$f(x) = x^2e^{x^3} = \left(\frac{1}{3}\frac{du}{dx}\right)e^u$$

If our function-derivative pair is off by at most a constant factor, we can still proceed, thanks to the anti-constant-multiple rule. Here is how:

$$\begin{aligned}
\int x^2 e^{x^3}\,dx &= \int e^{x^3} \cdot \left[x^2\right] dx \\
&= \int e^{x^3} \cdot \left[\left(\frac{1}{3}\right)\left(3x^2\right)\right] dx \\
&= \int e^{u} \cdot \left[\frac{1}{3}\frac{du}{dx}\right] dx \\
&= \frac{1}{3}\int e^{u} du \\
&= \frac{1}{3}e^{u} + C \\
&= \frac{1}{3}e^{x^3} + C
\end{aligned}$$

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Problem 12. Use the anti-chain rule to determine the antiderivative family for $f(x) = x^3 \cos\left(x^4 - 1\right)$.

Example 12 *Use the anti-chain rule to determine the antiderivative family for* $f(x) = \dfrac{x-1}{x^2-2x+4}$.

Solution. In this case, if we let $u = x^2 - 2x + 4$, then $\frac{du}{dx} = 2(x-1)$. Therefore, then anti-chain rule tells us

$$\begin{aligned}
\int \frac{x-1}{x^2-2x+4} dx &= \int \frac{1}{x^2-2x+4}\left[x-1\right] dx \\
&= \int \frac{1}{x^2-2x+4}\left[\left(\frac{1}{2}\right)\left[2(x-1)\right]\right] dx \\
&= \int \frac{1}{u}\left[\left(\frac{1}{2}\right)\frac{du}{dx}\right] dx \\
&= \left(\frac{1}{2}\right)\int u^{-1} du \\
&= \left(\frac{1}{2}\right)\ln|u| + C \\
&= \left(\frac{1}{2}\right)\ln\left|x^2 - 2x + 4\right| + C
\end{aligned}$$

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Problem 13. Use the anti-chain rule to determine the antiderivative family for $f(x) = \dfrac{\sin(x)}{\cos(x)}$.

HOMEWORK:

Use the anti-chain rule to determine the antiderivative family for the following functions.

(1) $h(x) = 3\cos(3x)$

(2) $g(x) = -5x^3 \sin(x^4 + 4)$

(3) $f(x) = 2xe^{x^2+3}$

(4) $h(x) = (3x^2 + 1)(x^3 + x + 3)^6$

(5) $g(x) = \sec^2(x)\cos(4\tan(x))$

(6) $f(x) = \dfrac{\cos(x)}{\sin(x)}$

(7) $h(x) = \dfrac{1 + 2x}{x^2 + x + 3}$

(8) $g(x) = \left(x^2 + 1\right)\sin\left(x^3 + 3x\right)$

(9) $f(x) = \dfrac{3}{(4x - 5)^2}$

(10) $h(x) = \dfrac{x - 1}{\sqrt{x^2 - 2x + 7}}$

(11) $g(x) = \dfrac{\ln(\sqrt{x})}{\sqrt{x}}$

(12) $f(x) = \dfrac{\ln(x)}{x}$

Answers for last homework set

(1) $\int 3\cos(3x)dx = \sin(3x) + C$

(2) $\int \left[-5x^3 \sin(x^4 + 4)\right] dx = \frac{5}{4}\cos(x^4 + 4) + C$

(3) $\int 2xe^{x^2+3}dx = e^{x^2+3} + C$

(4) $\int (3x^2 + 1)(x^3 + x + 3)^6 dx = \dfrac{(x^3 + x + 3)^7}{7} + C$

(5) $\int \sec^2(x)\cos(4\tan(x))dx = \frac{1}{4}\sin(\tan(x)) + C$

(6) $\int \dfrac{\cos(x)}{\sin(x)}dx = \ln|\sin(x)| + C$

(7) $\int \dfrac{1 + 2x}{x^2 + x + 3}dx = \ln\left|x^2 + x + 3\right| + C$

(8) $\int \left(x^2 + 1\right)\sin\left(x^3 + 3x\right) dx = -\frac{1}{3}\cos\left(x^3 + 3x\right) + C$

(9) $\int \dfrac{3}{(4x - 5)^2}dx = -\dfrac{3}{4\,(4x - 5)} + C$

(10) $\int \dfrac{x - 1}{\sqrt{x^2 - 2x + 7}}dx = \sqrt{x^2 - 2x + 7} + C$

(11) $\int \dfrac{\ln(\sqrt{x})}{\sqrt{x}}dx = 2\left(\sqrt{x}\ln(\sqrt{x}) - \sqrt{x}\right) + C$

(12) $\int \dfrac{\ln(x)}{x}dx = \dfrac{(\ln(x))^2}{2} + C$