CONSTRAINED OPTIMIZATION

In a *constrained optimization problem*, we seek to find the maximum or minimum output of a certain function subject to restrictions. The function in question is called the *optimization function*, and it frequently contains more than

one variable. The restrictions are called *constraints*, and they are used to reduce the number variables in the formula for the function and to establish relevant restrictions on its domain.

Every constrained optimization problem is solved the same way.

1. First, you identify the changing quantities in the problem and assign variables to represent the possible values of these quantities.

2. Second, you develop a formula that relates the values of the changing quantities. This formula is your *optimization formula.*

3. Third, you identify the constraints given in the problem and use them to reduce the number of variables in the optimization formula until you have a function of one variable. This function is called the *optimization function.*

4. Fourth, you use the constraints to determine the relevant domain of the optimization function.

5. Finally, you *optimize* the function; that is, you determine the values of the input variable that produce the maximum or minimum output of the function (whichever is required) within the relevant domain.

It is often possible to optimize the function by graphing it on a graphing calculator, using the relevant domain to set the viewing window. However, this method does not always produce a sufficiently accurate answer; and sometimes it is not easy to tell exactly where the desired extremum occurs. We can always use calculus to help us optimize.

Example 1 *Find two positive integers whose sum is equal to* 60 *and whose product is a large as possible.*

Solution. We have three changing quantities in this problem, namely the two integers, and their product. We begin by assigning variables to each of these quantities.

- Let m be one of the two integers, and let n be the other.
- Let P represent the product of these two integers.

The phrase "as large as possible" is associated with the variable P; hence, the problem tells us that *we are optimizing the product of the two integers.* Our optimization formula is therefore

$$P = mn$$

Now, the formula relating P and the integers m and n contains three variables; in particular, the output P depends on the values of two inputs, namely m and n. At this point, we return to the statement of the problem to look for constraints. Constraints usually take the form of inequalities or equations involving constants.

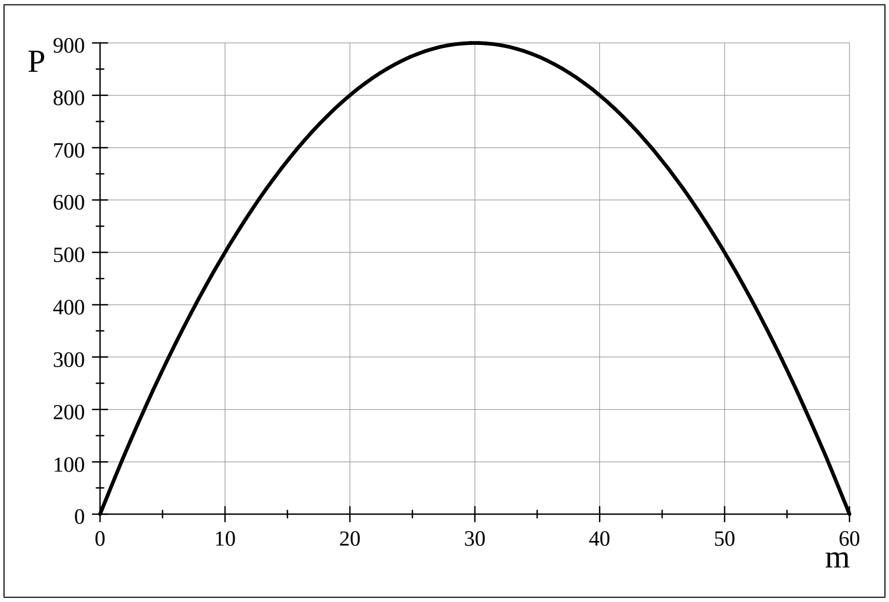
- We are told that m and n must be positive. This gives us two inequalities, namely $0 < m$ and $0 < n$. Inequalities are usually used in determining the relevant domain of the optimization function.

- We are told that the sum of m and n must be 60. This gives us the equation $60 = m + n$. Equations like this are used to reduce the number of variables.

We can solve the equation $60 = m + n$ for one of the variables and use substitute the result for that variable in the optimization formula. For example,

$$60 = m + n \Longrightarrow 60 - m = n \qquad \text{so} \qquad P = mn \text{ can be written as } P = f(m) = m(60 - m)$$

The function f is our optimization function. The final step before optimizing is to determine the relevant domain for f. The input variable is m, so the relevant domain amount to restrictions on the values of m. We know that $0 < m$ since m must be positive. Now, the fact that n must also be positive tells us that we must have $m < 60$ as well, since $n = 60 - m$. The relevant domain for f is therefore $0 < m < 60$. Here is a sketch of the optimization function on the relevant domain.



The sketch shows us that the maximum value of the product is $P = 900$, and it occurs when $m = 30$. Now, since $n = 60 - m$, we know that $n = 30$ as well.

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We could have used calculus to optimize the function $P = f(m) = m(60 - m)$. We know that

$$f(m) = 60m - m^2 \qquad f'(m) = 60 - 2m \qquad f''(m) = -2$$

Setting $f'(m) = 0$ tells us that $m = 30$ is the only critical number for f; and since $f''(30) = -2 < 0$, the Second Derivative Test tells us that f has a relative maximum output at $m = 30$. Since $m = 30$ lies in the relevant domain, we may conclude that the product is maximized when $m = 30$.

Problem 1. Apply the methods from Example 1 to determine two positive numbers whose difference is 1000 and whose product is as small as possible.

Example 2 *Find the dimensions of a rectangle whose area is fixed at* 1000 *square inches and whose perimeter is as small as possible.*

Solution. In this case, we are working with three changing quantities, namely the dimensions (length and width) of the rectangle and its perimeter.

- Let P represent the possible values for the perimeter of the rectangle, measured in inches.
- Let x represent the possible values for the width of the rectangle, measured in inches.
- Let y represent the possible values for the length of the rectangle, measured in inches.

The problem tells us that we want the perimeter to be "as small as possible." This means we are optimizing the perimeter; and since we obtain the perimeter of a rectangle by adding up the lengths of all its sides, our optimization formula will be

$$P = 2x + 2y$$

The perimeter values depend on the values of two input variables; we must use the constraints in the problem to reduce this dependence to a single input variable. Now, the problem gives us only one restriction, namely

- The area of the rectangle is fixed at 1000 square inches.

Since x and y represent the dimensions of the rectangle, we know $1000 = xy$. This restriction allows us to represent one variable in terms of another. For example, we know

$$y = \frac{1000}{x}$$

Substituting this expression in place of y in the optimization formula allows us to recast that formula so that the values of P depend only on the values of x. This gives us our optimization function.

$$P = f(x) = 2x + \frac{2000}{x} = \frac{2x^2 + 2000}{x}$$

We need to determine the relevant domain for this function. Clearly we cannot have $x < 0$, since the values of x are lengths. We cannot have $x = 0$ either, since this value for x makes f undefined. If there are any other restrictions we need to place on x, they would have to come from the relationship between x and y and the fact that we must have $y > 0$. However, as long as $x > 0$, the formula telling us how y and x are related will produce positive values for y. Therefore, *there are no other restrictions on* x. The relevant domain is the ray $0 < x < +\infty$.

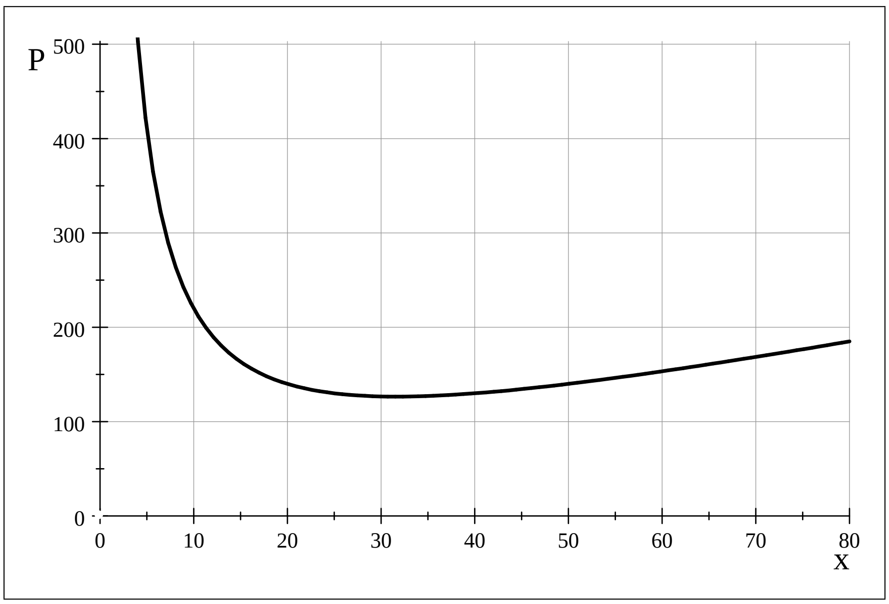
Having an unbounded relevant domain is unfortunate, since it does not give us a viewing window for graphing. Relying solely on the graphing calculator would therefore require us to guess how big our viewing window should be, and this might cause us to miss the input value where the minimum occurs. This is a situation where calculus can help. Observe that

$$f'(x) = 2 - \frac{2000}{x^2} = \frac{2x^2 - 2000}{x^2}$$

The derivative function will be undefined when $x = 0$ and will have output 0 when

$$2x^2 - 2000 = 0$$

The last equation is satisfied when $x = \pm\sqrt{1000}$; that is, when $x \approx \pm 31.62$. Of these three critical numbers for f, only $x = \sqrt{1000}$ lies in the relevant domain. Therefore, we would expect the minimum perimeter to occur when $x = \sqrt{1000}$. This information tells us what our viewing window should be — it should be large enough to let us see what the graph is doing at $x \approx 31.62$.



Based on the behavior of the graph shown above *along with the fact that we know f has a critical number at $x \approx 31.62$* tells us that the minimum perimeter does indeed occur when the rectangle has a width of $x = \sqrt{1000}$ inches. The corresponding length would be

$$y = \frac{1000}{\sqrt{1000}} = \sqrt{1000} \text{ inches}$$

Notice that the perimeter is minimized when the rectangle is a square.

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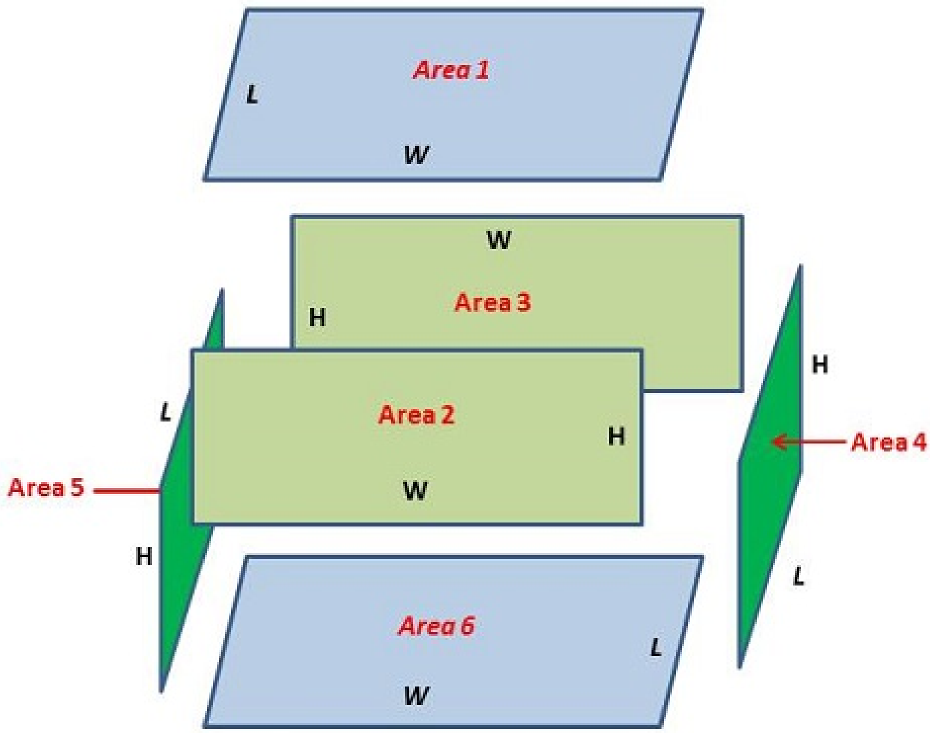
Problem 2. Find the dimensions of a rectangle whose perimeter is fixed at 100 feet and whose area is as large as possible.

Example 3 *A manufacturing company needs to make a closed rectangular box whose volume is fixed at* 10 *cubic meters. Shipping needs dictate that the length of the box must be twice its width. If the material for the base costs \$10.00 per square meter while material for the sides and top cost \$6.00 per square meter, what dimensions will minimize the cost of making the box?*

Solution. In this problem, we have four varying quantities.

- Let C represent the possible values for the cost of the manufacturing the box, measured in dollars.
- Let W represent the width of the box, measured in meters.
- Let L represent the length of the box, measured in meters.
- Let H represent the height of the box, measured in meters.

The problem tells us that we want to minimize the cost of the box. Our optimization function must therefore relate the dimensions of the box to the cost of making it. There will be six faces to the box — four sides, a top, and a bottom. Since the material cost is given per unit of area, the cost of manufacturing each face is determined by the area of that face. A diagram helps organize what we know.



The total cost of manufacturing the box will be the sum of the costs for manufacturing each face. The side faces cost \$6.00 per area unit, while the top and bottom faces cost \$10.00 per area unit. Therefore, our optimization formula will be

$$\begin{aligned} C &= 10(\text{Area } 1 \ + \ \text{Area } 6) + 6(\text{Area } 2 \ + \ \text{Area } 3 + \ \text{Area } 4 \ + \ \text{Area } 5) \\ &= 10(WL + WL) + 6(WH + WH + LH + LH) \\ &= 20WL + 12WH + 12LH \end{aligned}$$

The cost values depend on the values of three input variables. We must use constraints to reduce this dependence to a single variable. The problem gives us the following restrictions.

- The volume of the box is fixed at 10 cubic meters; that is, $10 = WHL$.
- The length of the box is twice the width; that is, $L = 2W$.

We can use the first restriction to write H in terms of L and W; in particular, we know $H = \frac{10}{LW}$. Applying this relationship to the cost formula gives us

$$\begin{aligned} C &= 20WL + 12W\left(\frac{10}{LW}\right) + 12L\left(\frac{10}{LW}\right) \\ &= 20WL + \frac{120}{L} + \frac{120}{W} \end{aligned}$$

The second restriction tells us how L depends on W. Applying this relationship to the cost formula gives us

$$\begin{aligned} C = f(W) &= 20W(2W) + \frac{120}{2W} + \frac{120}{W} \\ &= 20W^2 + \frac{180}{W} \\ &= \frac{20W^3 + 180}{W} \end{aligned}$$

Clearly, we must have $W > 0$. There is nothing in the relationship between H and W or between L and W to suggest any other restrictions on the domain of f. Therefore, the relevant domain for f is the ray $0 < W < +\infty$.

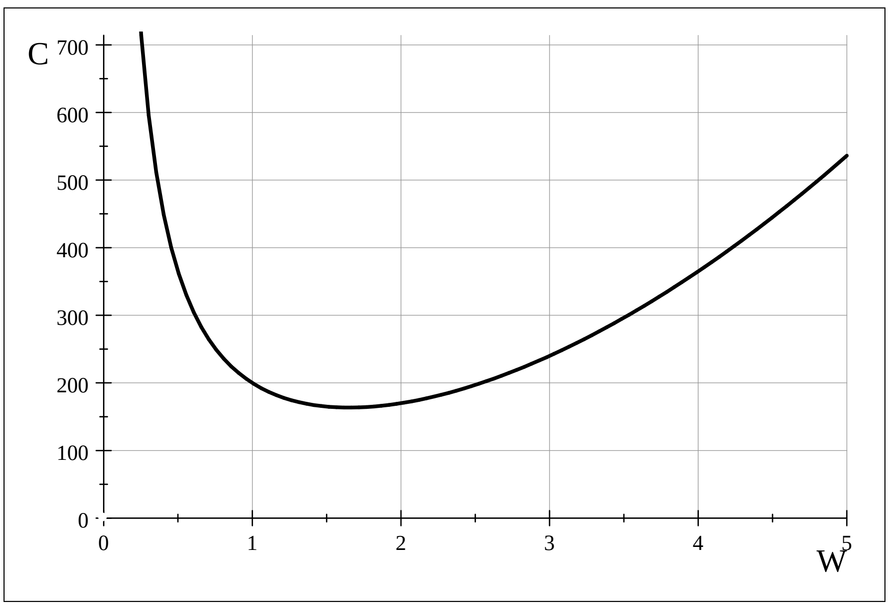
Once again, we will need calculus to help us determine what value of the width gives us the mimimum cost. Observe that

$$f'(W) = 40W - \frac{180}{W^2} = \frac{40W^3 - 180}{W^2}$$

The derivative will be undefined when $W = 0$. The derivative will have output 0 when $40W^3 - 180 = 0$. Now,

$$40W^3 - 180 = 0 \Longrightarrow W = \sqrt[3]{4.5}$$

Therefore, the only critical number for f in the implied domain is $W \approx 1.65$ meters. We would expect that the minimum cost will occur for this value of the width. Let's take a look at the graph of f to see if our expectation is correct.



It is clear that the minimum cost does indeed occur when $W = \sqrt[3]{4.5}$. The other dimensions will be $L = 2\sqrt[3]{4.5}$ meters and $H = \dfrac{5}{\sqrt[3]{20.25}}$ meters.

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HOMEWORK: Section 4.7 Page 337, Problems 2, 3, 4, 9, 14, 15, 17