USING DEFINITE INTEGRALS

Part I

Area Between Curves

In these notes, we will explore several applications of the definite integral that arise by exploiting its definition as the limit of Riemann sums. We can use definite integrals to model some process or determine some quantity whenever it is possible to imagine breaking the process or quantity into numerous small parts where changing values are "almost" constant. As a first example, let's consider how we can determine the area enclosed between two integrable curves. The key to the method lies in remembering that, for a function f defined on the interval between $x = a$ and $x = b$, the definite integral of f is defined by

$$\int_a^b f(x)dx = \lim_{n \longrightarrow \infty} \sum_{j=0}^{n-1} f(x_j^*)\Delta x_j$$

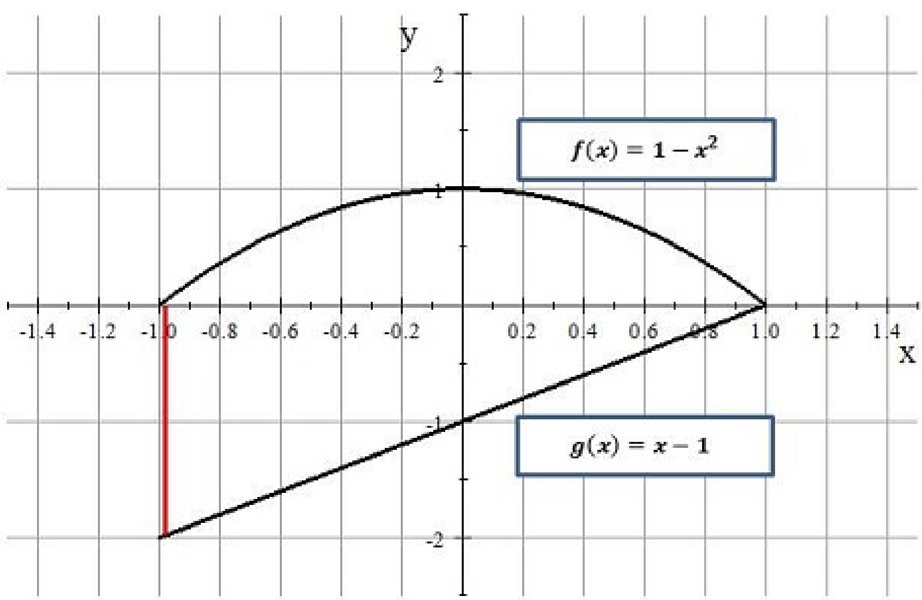
where n represents the number of subintervals (of equal width) you have divided the interval into, x_j^* represents an input value chosen at random from the jth subinterval, and Δx_j represents the (signed) width of the jth subinterval. In this case, the signed width is always the same —

$$\Delta x_j = \frac{b-a}{n}$$

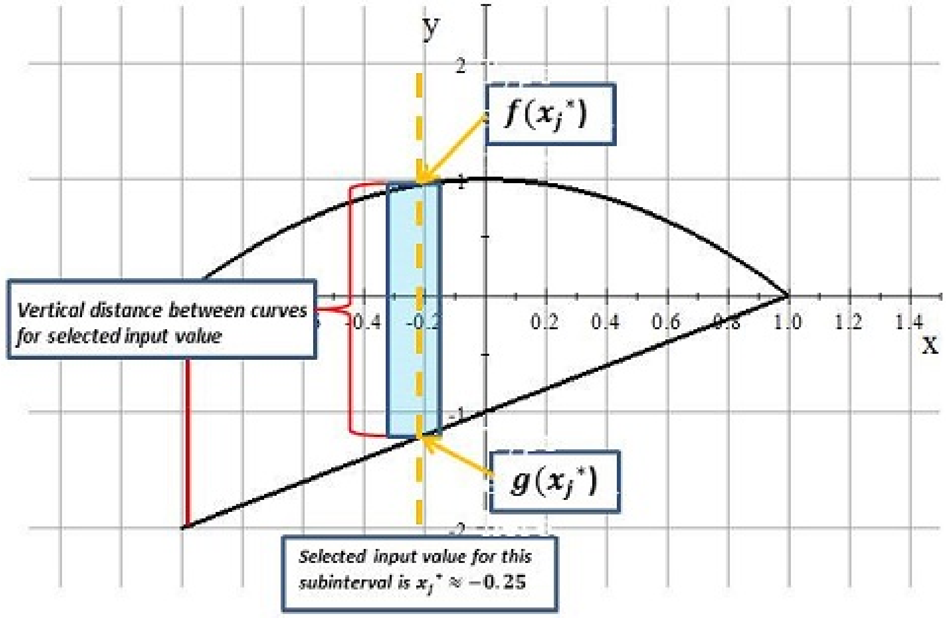
It is not necessary for the signed widths of the subintervals to be the same, it just makes defining the limit process trickier if they are not.

Example 1 *Use a definite integral to determine the area enclosed between the curves* $f(x) = 1 - x^2$ *and* $g(x) = x - 1$ *on the interval* $-1 \le x \le 1$.

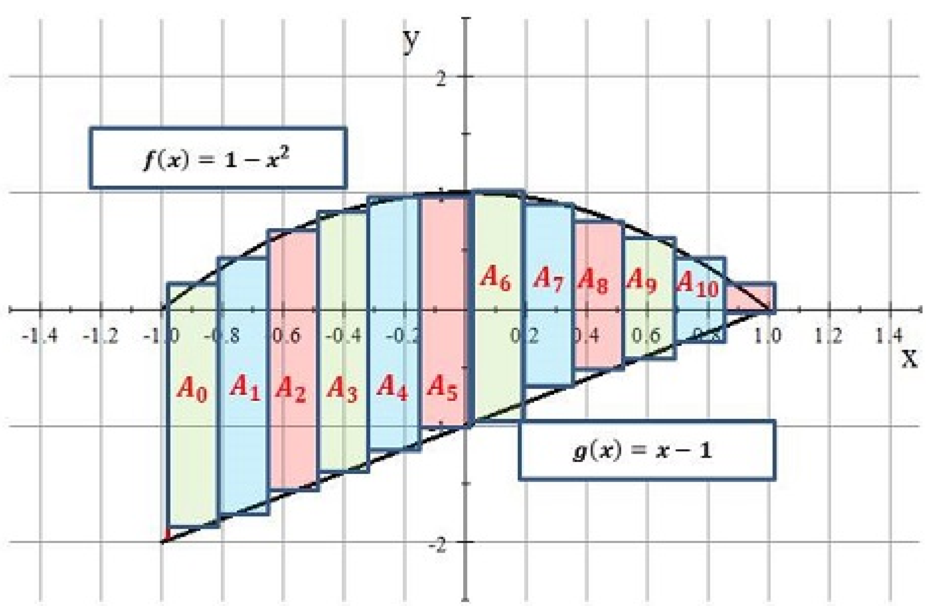
Here is a graph of the two functions showing the area we want to compute.



We can approximate the area between these curves using specially chosen rectangles. Suppose we divide the interval $[-1, 1]$ into n subintervals of equal width. Consider any one of these subintervals, say the interval I_j. At any input value x_j^* in this interval, the vertical distance between the curves f and g is given by $f(x_j^*) - g(x_j^*)$. We can think of this vertical distance as the height of a rectangle whose width is the width of the subinterval I_j.



The width of the subinterval I_j is $\Delta x_j = \dfrac{2}{n}$. With this in mind, the area of the shaded rectangle above is $A_j = (f(x_j^*) - g(x_j^*))\Delta x_j$. If we add up the areas of all n shaded rectangles, we obtain an approximation to the actual area enclosed between the curves. For example, here is what the approximation would look like if we divided $[-1, 1]$ into twelve subintervals of equal width and randomly selected an input value from each subinterval to determine the height of corresponding rectangles.



$$
\begin{aligned}
\text{Area} &\approx A_0 + A_1 + \dots + A_{10} + A_{11} \\
&= [f(x_0^*) - g(x_0^*)]\left(\frac{2}{12}\right) + [f(x_1^*) - g(x_1^*)]\left(\frac{2}{12}\right) + \dots + [f(x_{11}^*) - g(x_{11}^*)]\left(\frac{2}{12}\right) \\
&= \sum_{j=0}^{11} (f(x_j^*) - g(x_j^*))\Delta x_j
\end{aligned}
$$

Now, if we divide the interval $[-1, 1]$ into more and more subintervals (so that the width of each subinterval gets smaller and smaller), this approximation to the actual area will get better and better. Consequently, we know

$$
\begin{aligned}
\text{Area} = \lim_{n \longrightarrow \infty} \sum_{j=0}^{n-1} (f(x_j^*) - g(x_j^*))\Delta x_j &\equiv \int_{-1}^{1} [f(x) - g(x)]dx \\
&= \int_{-1}^{1} \left[(1 - x^2) - (x - 1)\right] dx \\
&= \int_{-1}^{1} \left[2 - x - x^2\right] dx \\
&= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{x=-1}^{x=1} \\
&= \frac{10}{3}
\end{aligned}
$$

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AREA BETWEEN CURVES (Part 1)

Suppose that $\mathcal{R}$ is a region in the plane that is bounded above by the curve $y = A(x)$ and bounded below by the curve $y = B(x)$. Suppose further that the region is bounded to the left by the real number $x = a$ and to the right by the real number $x = b$. If A and B are integrable functions on the interval $[a, b]$, then the area of the region $\mathcal{R}$ is given by

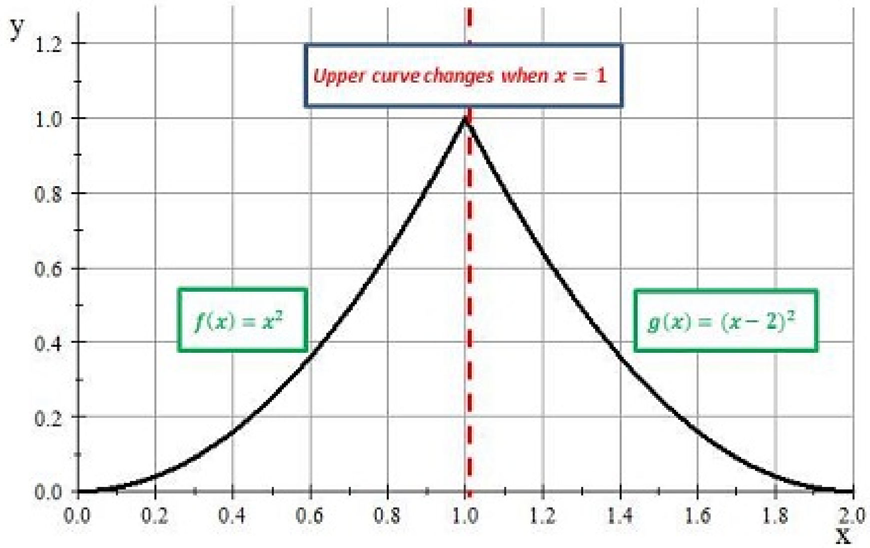
$$A_{\mathcal{R}} = \int_a^b [A(x) - B(x)]\, dx$$

Problem 1. Determine the exact area of the region enclosed between the curves $f(x) = 2 + x$ and $g(x) = 1 - 2x$ on the interval $-1 \le x \le 2$.

Problem 2. Determine the exact area of the region enclosed between the curves $f(x) = x^3$ and $g(x) = x$ on the interval $0 \le x \le 2$.

Example 2 *Determine the area enclosed between the curves* $f(x) = x^2$, $g(x) = (x-2)^2$ *and* $h(x) = 0$ *on the interval* $0 \leq x \leq 2$.

Solution. First, we must sketch the region so that we can decide which curve serves as the upper bound and which curve serves as the lower bound for the region.



According to the diagram, we see that the lower curve is $y = B(x) = 0$. However, the role of upper curve changes at $x = 1$; consequently, we will have to use two definite integrals to determine the area. In particular,

$$\begin{aligned}
A &= \int_0^1 \left[x^2 - 0\right] dx + \int_1^2 \left[(x-2)^2 - 0\right] dx \\
&= \int_0^1 x^2 dx + \int_1^2 (x-2)^2 \, dx \\
&= \left.\frac{x^3}{3}\right|_{x=0}^{x=1} + \left.\frac{(x-2)^3}{3}\right|_{x=1}^{x=2} \\
&= \frac{1}{3}\left([1-0] + [0-(-1)]\right) \\
&= \frac{2}{3}
\end{aligned}$$

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There is another way we could compute the area of the region from the previous example — one that allows us to avoid having to evaluate two definite integrals. Consider the curves f and g as functions of y *instead of* functions of x:

$$y = f(x) = x^2 \Longrightarrow \sqrt{y} = f^{-1}(y) = x \quad (\text{for } x \geq 0 \text{ only})$$

$$y = g(x) = (x-2)^2 \Longrightarrow 2 - \sqrt{y} = g^{-1}(y) = x \quad (\text{for } x \leq 2 \text{ only})$$

When these curves are treated as functions of y, notice that f^{-1} is the left bounding curve of the region, and g^{-1} is the right bounding curve for the region ... as long as we are in the interval $0 \leq y \leq 1$. Furthermore, these roles do not change for the region. Therefore, *if we integrate with respect to* y, we do not have to break up the region. Observe

$$\begin{aligned} A &= \int_{y=0}^{y=1} [(2-\sqrt{y})-\sqrt{y}]\,dy \\ &= \int_0^1 (2-2\sqrt{y})\,dy \\ &= \left[2y-\frac{4}{3}y^{3/2}\right]_{y=0}^{y=1} \\ &= 2-\frac{4}{3} \\ &= \frac{2}{3} \end{aligned}$$

AREA BETWEEN CURVES (Part 2)

Suppose that $\mathcal{R}$ is a region in the plane that is bounded on the left by the curve $x = L(y)$ and bounded on the right by the curve $x = R(y)$. Suppose further that the region is bounded below by the real number $y = a$ and above by the real number $y = b$. If L and R are integrable functions on the interval $[a, b]$, then the area of the region $\mathcal{R}$ is given by

$$A_{\mathcal{R}} = \int_a^b [R(y) - L(y)]\,dy$$

Problem 3. Consider the region $\mathcal{R}$ enclosed between the curves $y = f(x) = x + 6$ and $y = g(x) = x^3$ on the interval $0 \le x \le 2$.

Part (a): Determine the area of this region by integrating with respect to x.

Part (b): Determine the area of this region by rewriting the bounding curves as functions of y and integrating with respect to y.

Problem 4. Determine the area of the regaion enclosed between the curves $y = f(x) = \ln(x)$, and $y = g(x) = x$ on the interval $1 \le x \le e$. (It is easier to integrate with respect to y.)

HOMEWORK: Section 6.1 Pages 434-435 Problems 1, 3, 5, 7, 13, 15, 16, 19, 20, 21, 24, 29, 31, 32, 35, 36

Part II

Volume by Slicing

A *right cylinder* is formed by taking any planar region $\mathcal{R}$ and translating the region a distance h in a direction perpendicular to $\mathcal{R}$. The region can have any shape. If $A_{\mathcal{R}}$ denotes the area of the region $\mathcal{R}$, then the volume of the cylinder will be

$$V = A_{\mathcal{R}} \cdot h$$

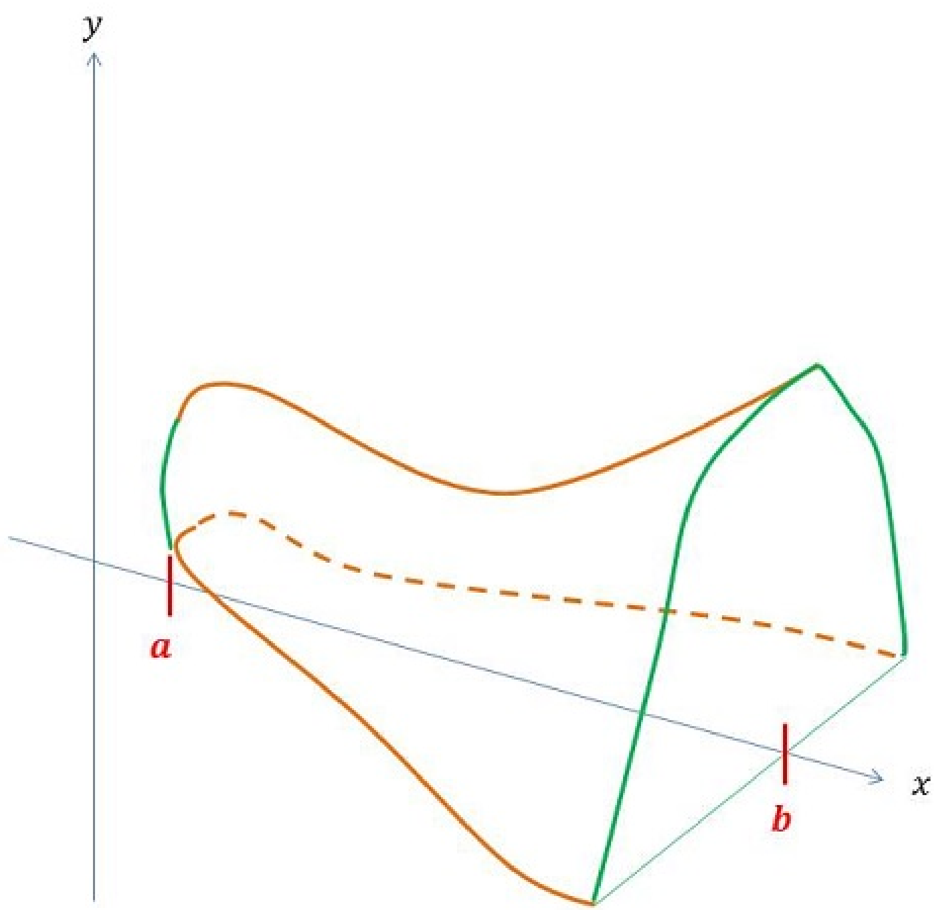
We can use right cylinders to help us compute the volume of many solids. Suppose that we can orient a solid $\mathcal{S}$ with respect to the xy-plane so that the following conditions are met:

- The solid casts a "shadow" on the x-axis that is a closed interval $a \leq x \leq b$.
- The cross sections of $\mathcal{S}$ *taken parallel to the y-axis* form an integrable function $y = A(x)$ on the interval $a \leq x \leq b$.

Under these conditions, the volume of the solid can be computed using a definite integral. In particular,

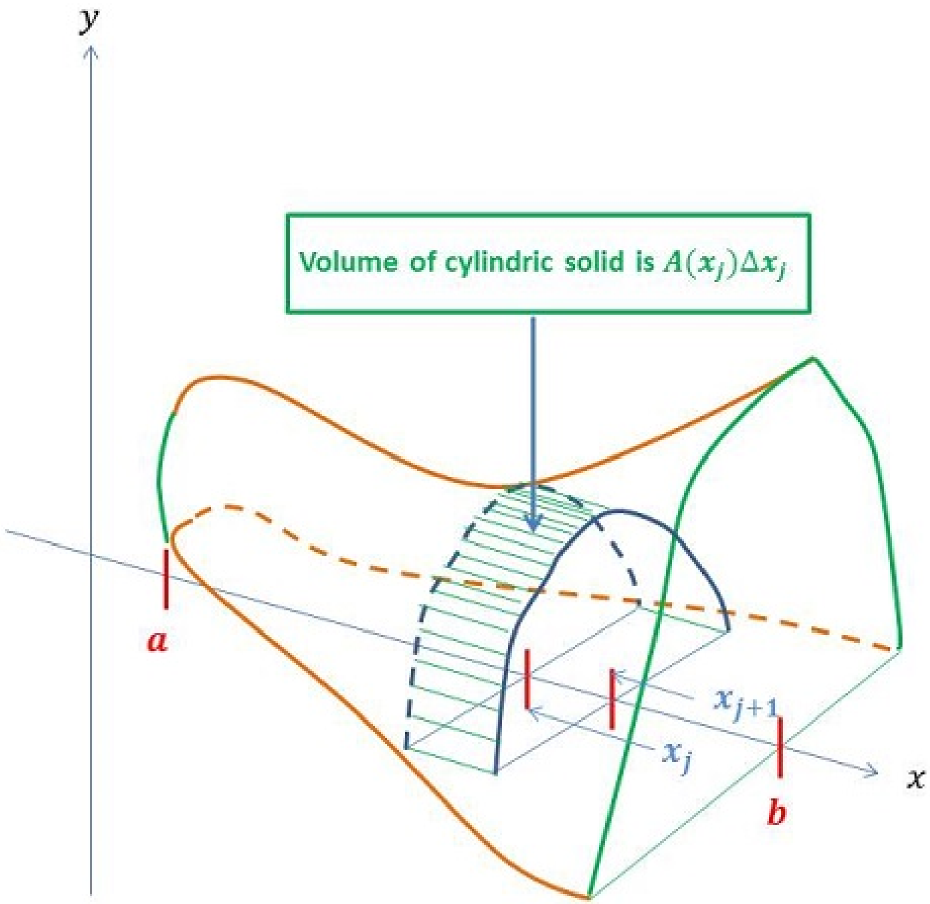
$$V_{\mathcal{S}} = \int_a^b A(x)dx$$

We can use the volume formula for right cylinders to understand why this formula is valid. To begin, consider the solid $\mathcal{S}$ shown below.



To approximate the volume of this solid, we will imagine dividing the interval $a \leq x \leq b$ into n subintervals of equal width (where n is very large). On each subinterval I_j (where j is any integer between

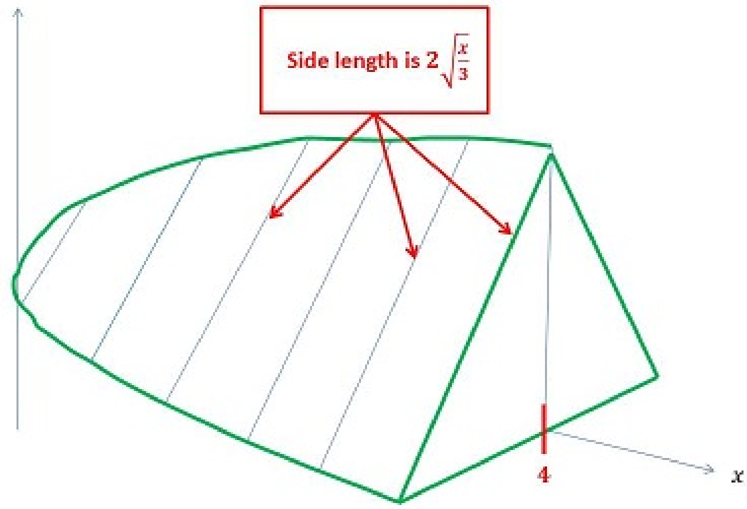
0 and $n-1$), we let the left endpoint be x_j and let the right endpoint be x_{j+1}. Now, imagine that we take a cross-section by slicing through the solid parallel to the y-axis at the point x_j. Let $A(x_j)$ represent the area of this cross-section. If we translate this cross section across the horizontal distance $\triangle x_j$ (the width of the subinterval I_j) we create a right cylinder as shown below.



If we sum up the volumes of the right cylinder created on each of the n subintervals, we have an approximation of the volume of the solid. Furthermore, if we divide up the interval $a \le x \le b$ into more and more subintervals (so that the widths of the subintervals gets smaller and smaller), the corresponding collection of right cylinders will approximate the solid better and better. Therefore, we know

$$V_{\mathcal{S}} = \lim_{n \longrightarrow \infty} \sum_{j=0}^{n-1} A(x_j) \triangle x_j = \int_a^b A(x)dx$$

Example 3 *Construct a formula for the volume of the solid whose cross section parallel to the y-axis at any value of x in the interval $0 \le x \le 4$ feet are equliateral triangles of height $\sqrt{x}$.*



Solution. The cross sections of this solid are equilateral triangles. For any value of x in the interval $0 \leq x \leq 4$ feet, the height of each triangle is $\sqrt{x}$ feet. The relationship between the sides of a $30° - 60° - 90°$ triangle tell us that the sides of the cross section at the value x will be $2\sqrt{x/3}$ feet in length. Consequently, the cross-sectional area function for this solid will be

$$A(x) = \left(\frac{1}{2}\right)\left(2\sqrt{\frac{x}{3}}\text{ ft}\right)\left(\sqrt{x}\text{ ft}\right) = \frac{x}{\sqrt{3}}\text{ft}^2$$

Therefore, the volume of this solid will be

$$V = \frac{1}{\sqrt{3}}\int_0^4 x dx = \left.\frac{x^2}{2\sqrt{3}}\right|_{x=0}^{x=4} = \frac{8}{\sqrt{3}}\text{ ft}^3$$

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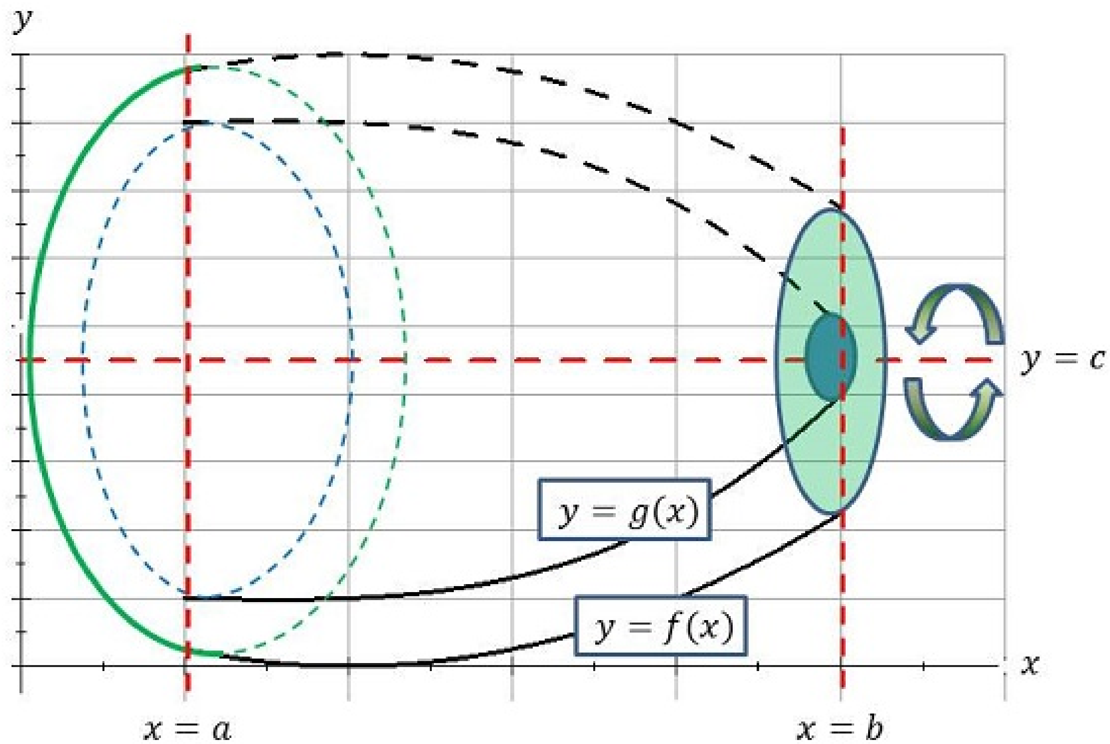
The "method of slicing" as this integration technique is often called, is typically used to compute the volume for a *solid of revolution*. A solid of revolution is created when the plane region enclosed between two curves f and g is revolved around a fixed line that is parallel to the input axis for the functions.

DEFINITION: Suppose that $y = f(x)$ and $y = g(x)$ are two continuous curves defined on the input interval $a \leq x \leq b$, and suppose that $y = c$ is a horizontal line. Suppose further that $g(x) - c \leq f(x) - c$ on the interval $a \leq x \leq b$. Let $\mathcal{R}$ represent the plane region enclosed between $y = f(x) - c$ and $y = g(x) - c$

on this interval. We obtain a *solid of revolution* by revolving $\mathcal{R}$ around the line $y = c$.



The diagram above shows the solid created when the region $\mathcal{R}$ enclosed between the curves $y = f(x)$ and $y = g(x)$ is revolved around the line $y = c$. Notice that cross sections of this solid taken *perpendicular to the axis of revolution* are all washers whose inner circle is traced out by the function g and whose outer circle is tracted out by the function f. In particular, at any input value x in the interval $a \leq x \leq b$, the area of the cross-sectional disc will be

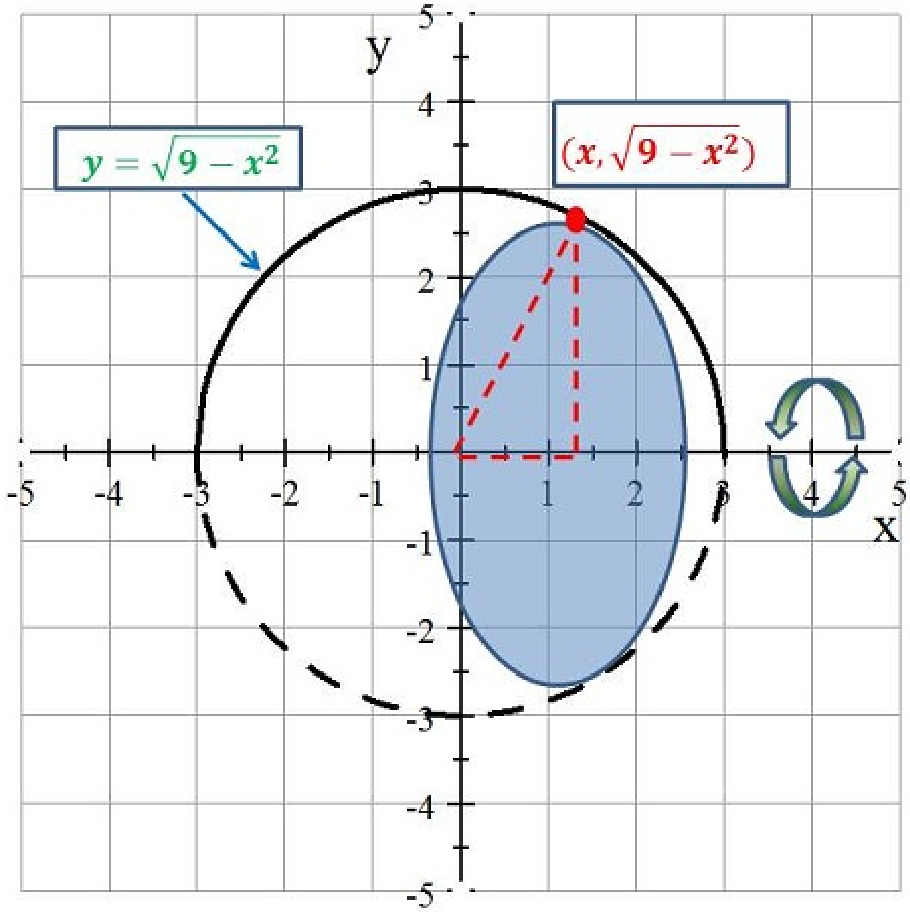
$$A(x) = \pi \left[(f(x) - c)^2 - (g(x) - c)^2 \right]$$

Since the functions f and g are continuous, we know they are integrable. Therefore, the volume of this solid will be given by the formula

$$V = \int_a^b A(x) dx = \pi \int_a^b \left[(f(x) - c)^2 - (g(x) - c)^2 \right] dx$$

Using this formula to compute the volume of a solid of revolution is often called using the *method of washers* since the cross sections used to create the definite integral are all washers. When the inner function g is equal to the line $y = c$, the technique is sometimes called the *method of discs* simply because there is no inner core hollowed out in the solid.

Example 4 *Use the Method of Discs to compute the volume of the sphere created by taking the region enclosed between the functions* $f(x) = \sqrt{9 - x}$, $g(x) = 0$ *and the constants* $x = -3$, $x = 3$ *and revolving it around the line* $y = 0$.



Solution. In this case, we know that

$$
\begin{aligned}
V &= \pi \int_{-3}^{3} \left[(f(x) - 0)^2 - (g(x) - 0)^2 \right] dx \\
&= \pi \int_{-3}^{3} \left[\left(\sqrt{9 - x^2} - 0 \right)^2 - (0 - 0)^2 \right] dx \\
&= \pi \int_{-3}^{3} \left(9 - x^2 \right) dx \\
&= \pi \left[9x - \frac{x^3}{3} \right]_{x=-3}^{x=3} \\
&= \pi \left[(27 - 9) - (-27 + 9) \right] \\
&= 36\pi
\end{aligned}
$$

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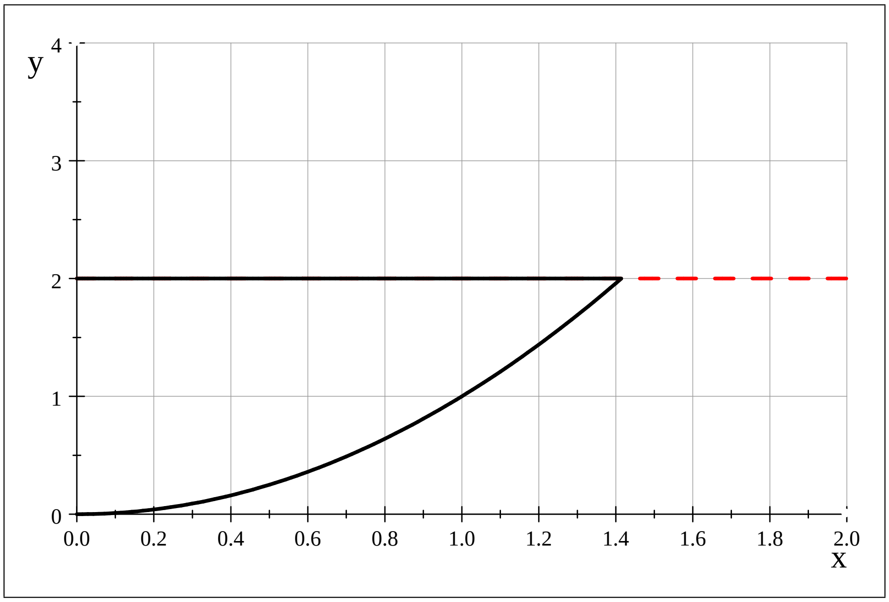
Example 5 *Use the Method of Discs to determine the volume of the solid obtained by revolving the region* $\mathcal{R}$ *around the line* $y = 2$*, where* $\mathcal{R}$ *denotes the region enclosed between the curves* $f(x) = x^2$ *and* $g(x) = 2$ *between* $x = 0$ *and* $x = \sqrt{2}$.



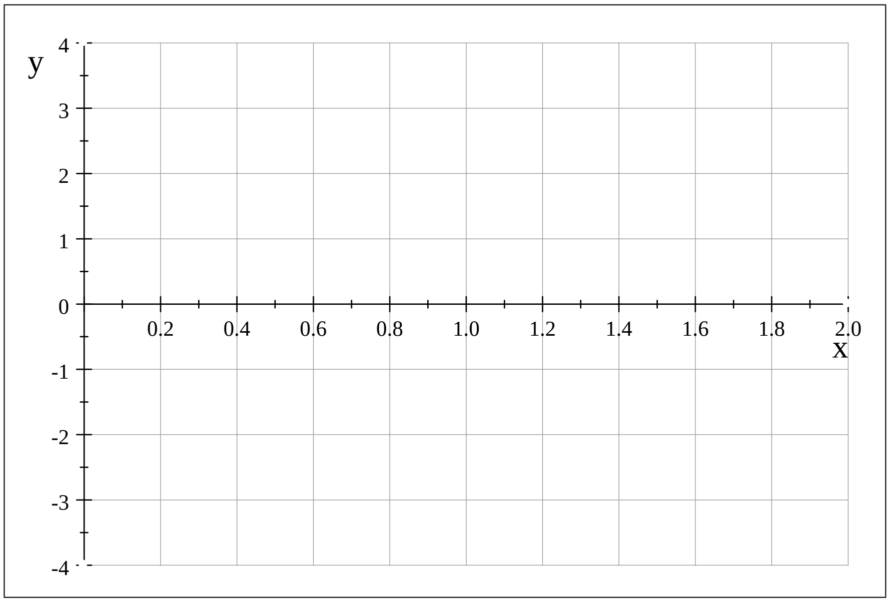
Solution. In this case, we know that the volume of the solid is given by

$$
\begin{aligned}
V &= \pi \int_0^{\sqrt{2}} \left[(f(x)-2)^2 - (g(x)-2)^2\right] dx \\
&= \pi \int_0^{\sqrt{2}} \left(x^2-2\right)^2 dx \\
&= \pi \int_0^{\sqrt{2}} \left(x^4-4x+4\right) dx \\
&= \pi \left[\frac{x^5}{5} - 2x^2 + 4x\right]_{x=0}^{x=\sqrt{2}} \\
&= \pi \left[\frac{4\sqrt{2}}{5} - 4 + 4\sqrt{2}\right] \\
&= \frac{\left(24\sqrt{2}-20\right)\pi}{5}
\end{aligned}
$$

****

Problem 1. Consider the plane region $\mathcal{R}$ enclosed between the curves $f(x) = e^x$ and $g(x) = 0$ between $x = 0$ and $x = 1$.

Part (a): Sketch the region $\mathcal{R}$ on the grid below.

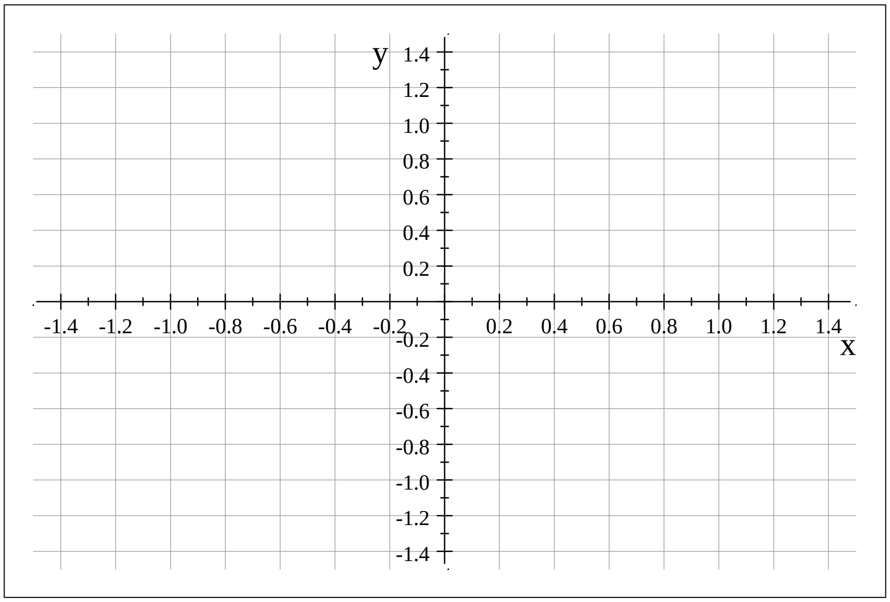


Part (b): Use the Method of Discs to compute the volume of the solid obtained by revolving $\mathcal{R}$ around the line $y = 0$.

Part (c): Use the Method of Washers to compute the volume of the solid obtained by revolving $\mathcal{R}$ around the line $y = 3$.

Problem 2. Consider the plane region $\mathcal{R}$ enclosed between the curves $f(x) = x$ and $g(x) = x^2$ between $x = -1$ and $x = 1$.

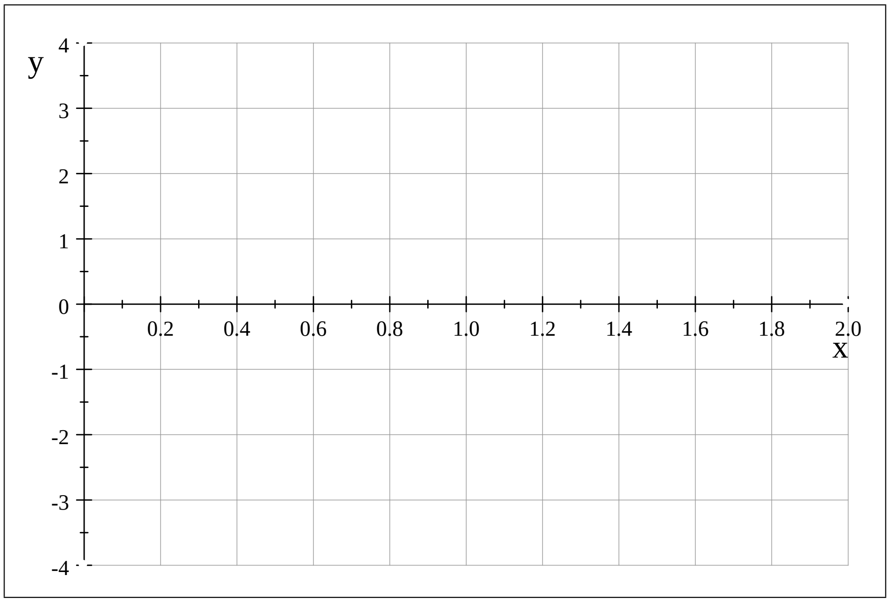
Part (a): Sketch the region $\mathcal{R}$ on the grid provided.



Part (b): Use the Method of Washsers to compute the volume of the solid obtained by revolving $\mathcal{R}$ around the line $y = 0$.

Problem 3. Consider the plane region $\mathcal{R}$ enclosed between the curves $f(x) = 2 - x^2$ and $g(x) = x^2$ between $x = 0$ and $x = 2$.

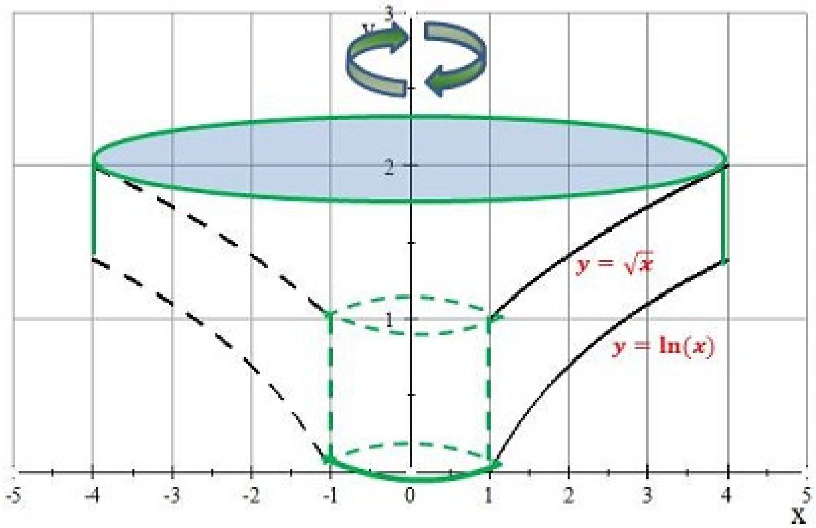
Part (a): Sketch the region $\mathcal{R}$ on the grid provided.



Part (b): Use the Method of Washers to compute the volume of the solid obtained by revolving $\mathcal{R}$ around the line $y = -2$.

HOMEWORK: Section 6.2 Pages 446 - 447 Problems 1, 2, 3, 11

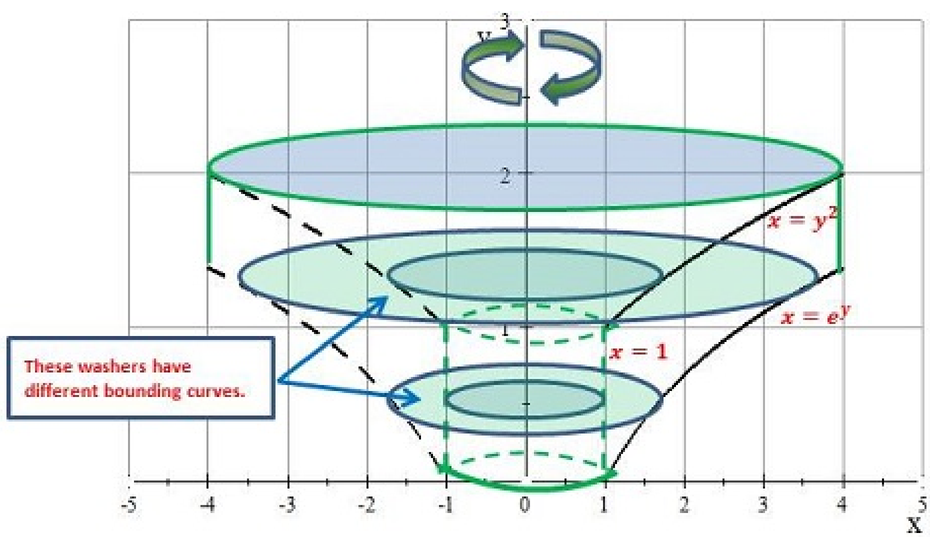
Example 6 *Let $\mathcal{R}$ be the plane region enclosed between the curves $f(x) = \ln(x)$ and $g(x) = \sqrt{x}$ between $x = 1$ and $x = 4$. Use the Method of Discs to compute the volume of the solid created by revolving $\mathcal{R}$ around the line $x = 0$.*



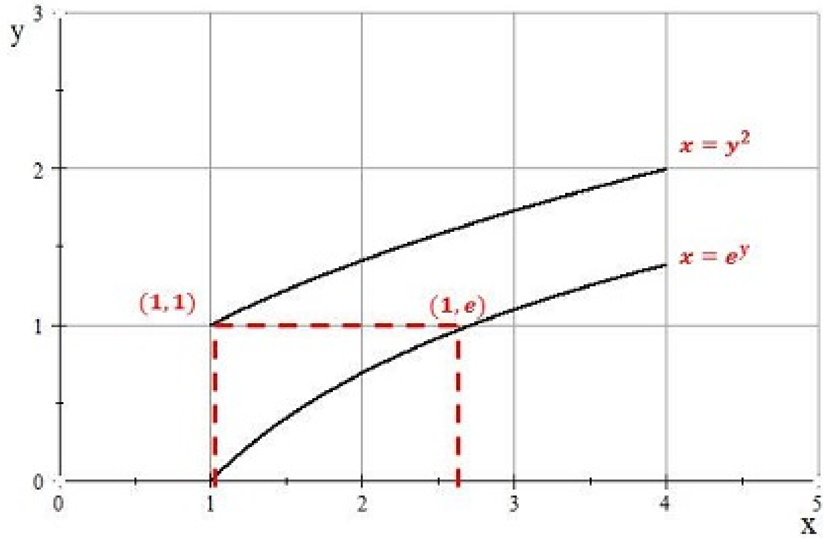
Solution. The Method of Washers requires that the washers be cross sections of the solid taken *perpendicular to the axis of revolution.* This means we must consider washers that are perpendicular to the y-axis. The bounding curves for these washers, while still the graphs of the functions f and g, will have to be rewritten as functions of y instead of functions of x. First, observe that

$$y = f(x) = \ln(x) \Longrightarrow e^y = f^{-1}(y) = x \qquad y = g(x) = \sqrt{x} \Longrightarrow y^2 = g^{-1}(y) = x \qquad \text{(For } x > 0\text{)}$$

Furthermore, when we construct washers perpendicular to the y-axis, we see that the process creates two sets of washers with different bounding curves. We will have to break up the integration process at the y-value where these families of washers change.



Let's take a look at the bounding curves $x = g^{-1}(y) = y^2$ and $x = f^{-1}(y) = e^y$. Looking at the graph, we can see that the inner (leftmost) bounding curve changes when $y = 1$. This tells us we must break the interval of integration at $y = 1$.



The volume of the solid is given by the formula

$$\begin{aligned}
V &= \pi \int_{y=0}^{y=1} \left[(e^y - 0)^2 - (1-0)^2 \right] dy + \pi \int_{y=1}^{y=2} \left[(e^y - 0)^2 - \left(y^2 - 0 \right)^2 \right] dy \\
&= \pi \int_0^1 \left(e^{2y} - 1 \right) dy + \pi \int_1^2 \left(e^{2y} - y^4 \right) dy \\
&= \pi \left[\frac{e^{2y}}{2} - y \right]_{y=0}^{y=1} + \pi \left[\frac{e^{2y}}{2} - \frac{y^5}{5} \right]_{y=1}^{y=2} \\
&= \pi \left[\left(\frac{e^2}{2} - 1 \right) - \left(\frac{1}{2} - 0 \right) \right] + \pi \left[\left(\frac{e^4}{2} - \frac{32}{5} \right) - \left(\frac{e^2}{2} - \frac{1}{5} \right) \right] \\
&= \pi \left(\frac{e^4}{2} - \frac{3}{2} - \frac{31}{5} \right) \\
&= \frac{5e^4 - 77}{10} \pi
\end{aligned}$$

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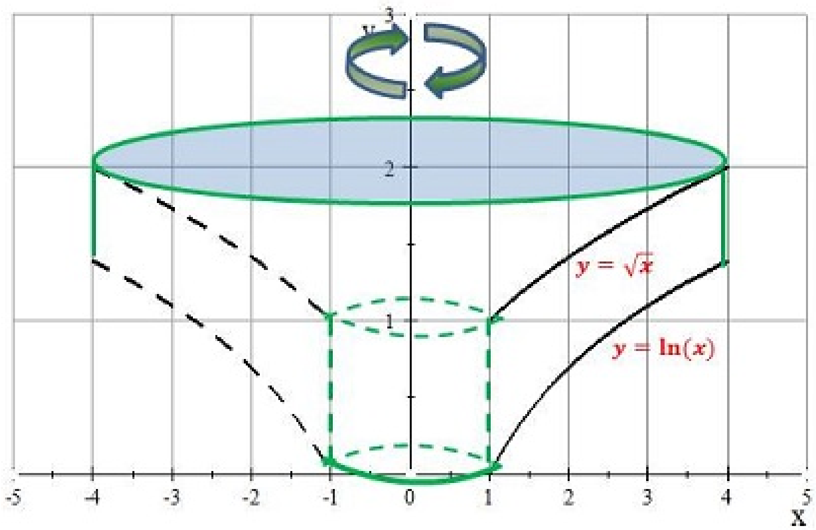
HOMEWORK: Section 6.2 Page 446, Problems 5, 9, 11, 15, 17

Part III

The Method of Shells

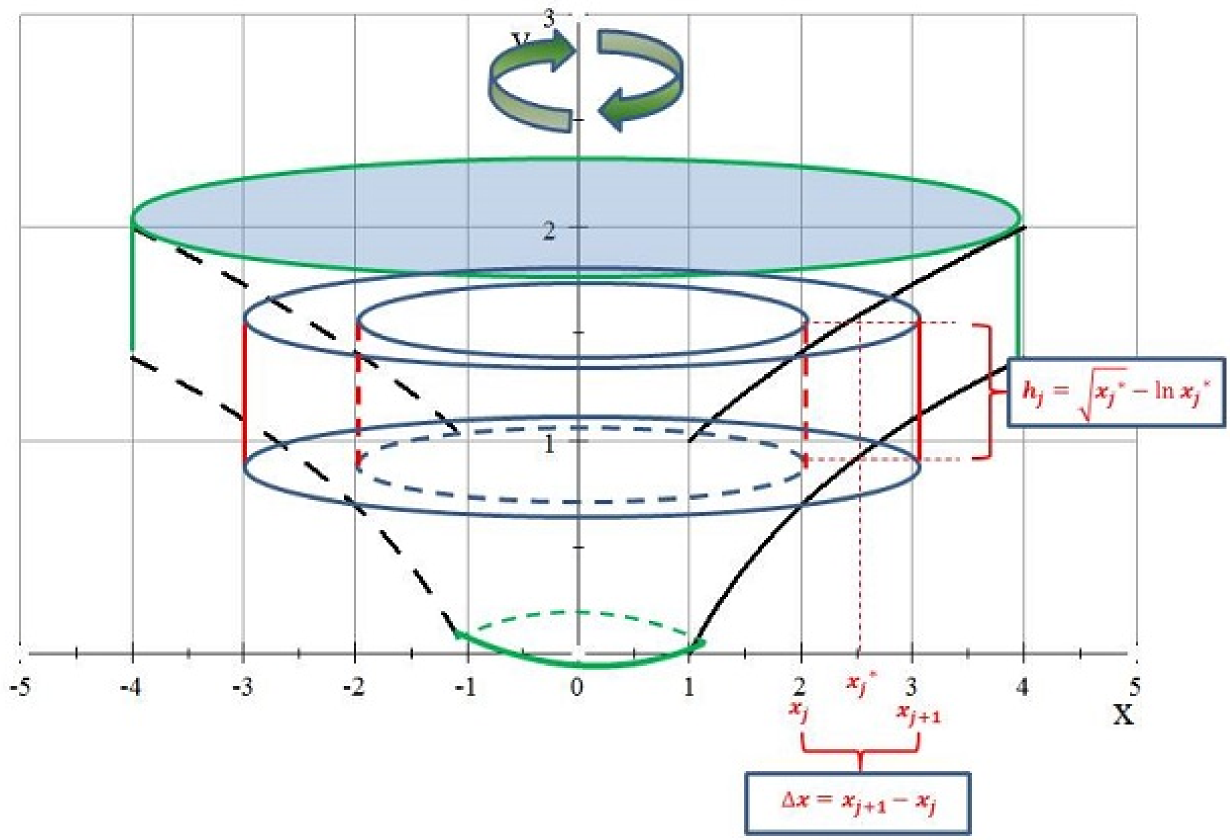
The Method of Washers can be used to compute the volume of most solids of revolution. However, the Method of Washers has one disadvantage — it requires that cross-sections of the solid be taken perpendicular to the axis of revolution. This can be a problem if the axis of revolution is parallel to the y-axis, since it forces us to rewrite the bounding curves as functions of y. In this section, we will introduce another method of computing volume that breaks ranks with the slicing methods introduced in the previous section.

Let's consider the solid introduced in the previous example.



When we used the Method of Washers, we took cross-sections perpedincular to the y-axis, and this forced us to rewrite the bounding curves as functions of y. This time, let's imagine dividing the interval $1 \leq x \leq 4$ into a large number (say n) of small subintervals, each of equal width $\triangle x$. Let I_j be any one of these n subintervals; and this time, imagine that we create a *cylindric shell* — on this interval. Let x_j^* denote the midpoint of the subinterval. We will let the height h of the shell be the vertical distance between the bounding curves at the midpoint — in other words, we let

$$h_j = g(x_j^*) - f(x_j^*) = \sqrt{x_j^*} - \ln(x_j^*)$$



The volume of this cylindric shell is given by the following formula:

$$\begin{aligned} V_j &= \text{[Volume of Outer Cylinder]} - \text{[Volume of Inner Cylinder]} \\ &= \pi h_j \left(x_{j+1}\right)^2 - \pi h_j \left(x_j\right)^2 \\ &= \pi \left[\left(x_{j+1}\right)^2 - \left(x_j\right)^2\right] h_j \end{aligned}$$

We can therefore approximate the volume of the solid by adding up the volumes of these n cylindric shells:

$$V \approx \sum_{j=0}^{n-1} \pi \left[\left(x_{j+1}\right)^2 - \left(x_j\right)^2\right] h_j$$

Unfortunately, the terms are not in the correct form for us to consider this approximation to be a *Riemann sum.* Therefore, we cannot take the limit of the current approximations to create a definite integral. Fortunately, we can use a little algebra to help out. Observe that

$$\begin{aligned} V_j &= \pi \left[\left(x_{j+1}\right)^2 - \left(x_j\right)^2\right] h_j \\ & \quad \pi \left[\left(x_{j+1} + x_j\right)\left(x_{j+1} - x_j\right)\right] h_j \\ &= \pi h_j \left(x_{j+1} + x_j\right) \triangle x \\ &= 2\pi h_j \left(\frac{x_{j+1} + x_j}{2}\right) \triangle x \\ &= 2\pi h_j x_j^* \triangle x \\ &= 2\pi x_j^* \left(\sqrt{x_j^*} - \ln(x_j^*)\right) \triangle x \end{aligned}$$

$$V = \lim_{n \to \infty} \sum_{j=0}^{n-1} 2\pi x_j^* \left(\sqrt{x_j^*} - \ln(x_j^*) \right) \triangle x = 2\pi \int_1^4 x \left(\sqrt{x} - \ln(x) \right) dx$$

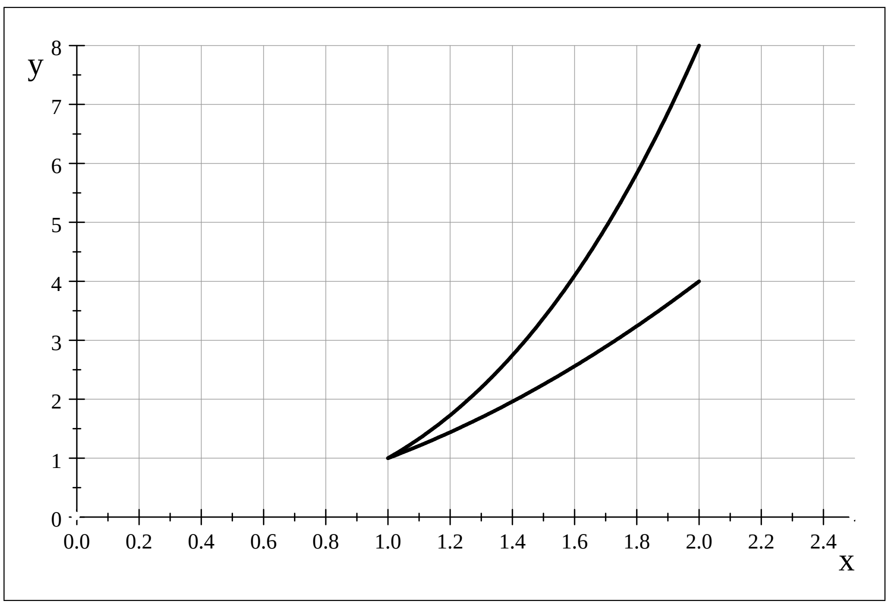
THE METHOD OF SHELLS

Suppose that $\mathcal{R}$ is a plane region enclosed between the continuous curves $y = f(x)$ and $y = g(x)$ on the interval $a \le x \le b$. If $f(x) \le g(x)$ on this interval, then the volume of the solid created by revolving $\mathcal{R}$ around the y-axis is given by the formula

$$V = 2\pi \int_a^b x \left(g(x) - f(x) \right) dx$$

The Method of Shells can be extended to cover solids created by revolving a plane region around any vertical line. However, care must be taken when setting up the formula as to where the vertical line is located with respect to the bounding curves.

Example 7 *Consider the plane region $\mathcal{R}$ enclosed between the curves $y = f(x) = x^2$ and $y = g(x) = x^3$ on the interval $1 \le x \le 2$. Find the volume of the solid obtained by revolving $\mathcal{R}$ around the y-axis. Use the Method of Shells, then use the Method of Washers.*



Solution. The diagram above shows the plane region $\mathcal{R}$. Notice that the graph of $g(x) = x^3$ lies above the graph of $f(x) = x^2$ on the interval $1 \le x \le 2$. Therefore, according to the Method of Shells, the volume of the solid will be

$$\begin{aligned} V &= 2\pi \int_1^2 x \left(x^3 - x^2 \right) dx \\ &= 2\pi \int_1^2 \left(x^4 - x^3 \right) dx \\ &= 2\pi \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_{x=1}^{x=2} \\ &= 2\pi \left[\left(\frac{32}{5} - 4 \right) - \left(\frac{1}{5} - \frac{1}{4} \right) \right] \\ &= \frac{49\pi}{10} \end{aligned}$$

Now, let's approach this problem using the Method of Washers. First, note that $y = f(x) = x^2$ implies $x = f^{-1}(y) = \sqrt{y}$ (when $x > 0$) and $y = g(x) = x^3$ implies $x = g^{-1}(y) = \sqrt[3]{x}$. Viewed from the y-axis, the region $\mathcal{R}$ must be broken into two subregions, because the rightmost bounding curve changes when $y = 4$. At this y-value, the rightmost bounding curve changes from $x = \sqrt{y}$ to $x = 2$. With this in mind, the Method of Discs gives us

$$
\begin{aligned}
V &= \pi \int_{y=1}^{y=4} \left[(\sqrt{y})^2 - (\sqrt[3]{y})^2 \right] dy + \pi \int_{y=4}^{y=8} \left[(2)^2 - (\sqrt[3]{y})^2 \right] dy \\
&= \pi \int_1^4 \left(y - y^{2/3} \right) dy + \pi \int_4^8 \left(4 - y^{2/3} \right) dy \\
&= \pi \left[\frac{y^2}{2} - \frac{3y^{5/3}}{5} \right]_{y=1}^{y=4} + \pi \left[4y - \frac{3y^{5/3}}{5} \right]_{y=4}^{y=8} \\
&= \pi \left[\left(8 - \frac{6\sqrt[3]{128}}{5} \right) - \left(\frac{1}{2} - \frac{3}{5} \right) \right] + \pi \left[\left(32 - \frac{96}{5} \right) - \left(16 - \frac{6\sqrt[3]{128}}{5} \right) \right] \\
&= \frac{49\pi}{10}
\end{aligned}
$$

****

It should be clear that the Method of Shells is considerably easier to use in the previous example. It is not always the case that the Method of Washers is more difficult to use than the Method of Shells, but this is often true working with solids created by revolving a region around a vertical line.

Problem 1. Consider the plane region $\mathcal{R}$ enclosed between the curves $y = f(x) = \sqrt[3]{x}$ and $y = g(x) = x$ on the interval $0 \leq x \leq 1$. Use the Method of Shells to compute the volume of the solid obtained by revolving $\mathcal{R}$ around the y-axis.

Problem 2. Consider the plane region $\mathcal{R}$ enclosed between the curves $y = f(x) = \sin(\pi x^2)$ and $y = f(x) = 1 + x$ on the interval $0 \leq x \leq \frac{\sqrt{2}}{2}$. Use the Method of Shells to compute the volume of the solid obtained by revolving $\mathcal{R}$ around the y-axis.

HOMEWORK: Section 6.3 Page 454, Problems 3, 4, 5, 6, 7, 9, 11, 13, 15, 17