RATIONAL FUNCTIONS AND INTEGRATION

A *rational function* is one whose output formula is the ratio of two polynomials. Using algebra and some trigonometric substitutions, it is possible to find the antiderivative family for any rational function. The process we use to accomplish this is based on the following antiderivative families

$$\int \frac{1}{mx+b}dx = \frac{1}{m}\ln|mx+b|+C \qquad (m \neq 0) \qquad\qquad \int \frac{x}{ax^2+b}dx = \frac{1}{2a}\ln\left|ax^2+b\right|+C \qquad (a, b \neq 0)$$

$$\int \frac{1}{(mx+b)^2+1}dx = \frac{1}{m}\arctan(mx+b)+C \qquad (m \neq 0)$$

All three of these antiderivative families are computed using simple substitutions.

A *proper* rational function is one in which the degree of the denominator exceeds the degree of the numerator. A rational function in which this is not the case is called an *improper* rational function.

PROPER RATIONAL FUNCTIONS

$$f(x) = \frac{1}{x} \qquad g(x) = \frac{x-1}{2x^2+x-1} \qquad h(x) = \frac{x^3-10x^2+x-3}{\sqrt{7}x^5+\pi x}$$

- Every improper rational function can be written as the sum of a polynomial and a *proper* rational function. This can be accomplished using polynomial division.

Example 1 *Use polynomial division to write* $f(x) = \dfrac{3x^4+3x^2-1}{x^2+x+1}$ *as the sum of a polynomial and a proper rational function.*

Solution. We use polynomial division to divide $d(x) = x^2+x+1$ into $n_1(x) = 3x^4+3x^2-1$.

STEP 1A: Ask "What must I multiply x^2 by in order to obtain $3x^4$?" ANSWER: I have to multiply by $3x^2$.

STEP 1B: Multiply every term of $d(x) = x^2+x+1$ by $3x^2$ and *subtract* the result from $n_1(x) = 3x^4-1$.

$$n_2(x) = 3x^4+3x^2-1-3x^2\left(x^2+x+1\right) = -3x^3-1$$

STEP 1C: Is the degree of the new polynomial $n_2$smaller than the degree of d? In this case, no — so repeat the process.

STEP 2A: Ask "What must I multiply x^2 by in order to obtain $-3x^3$?" ANSWER: I have to multiply by $-3x$.

STEP 2B: Multiply every term of $d(x) = x^2+x+1$ by $-3x$ and *subtract* the result from $n_2(x) = -3x^3-1$.

$$n_3(x) = -3x^3-1+3x\left(x^2+x+1\right) = 3x^2+3x-1$$

STEP 2C: Is the degree of the new polynomial $n_3$smaller than the degree of d? In this case, no — so repeat the process.

STEP 3A: Ask "What must I multiply x^2 by in order to obtain $3x^2$?" ANSWER: I have to multiply by 3.

STEP 3B: Multiply every term of $d(x) = x^2+x+1$ by $3x^2$ and *subtract* the result from $n_3(x) = 3x^2+3x-1$.

$$n_4(x) = 3x^2 + 3x - 1 - 3\left(x^2 + x + 1\right) = -4$$

STEP 3C: Is the degree of the new polynomial $n_2$smaller than the degree of d? In this case, yes — so the process stops.

$$f(x) = \frac{3x^4 - 1}{x^2 + x + 1} = 3x^2 - 3x + 3 - \frac{4}{x^2 + x + 1}$$

****

Now, the polynomial $d(x) = x^2+x+1$ is an *irreducible quadratic* — that is, it is a quadratic polynomial that has no real roots. Because of this fact, we can complete the square on this polynomial and thereby rewrite it in the form

$$d(x) = a\left(1 + u^2\right)$$

for an appropriate constant a and linear function $u(x) = mx + b$. To see how this is done, observe that

$$\begin{aligned}
x^2 + x + 1 &= \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} + 1 \qquad \text{(Take half of the } x\text{-coefficient, square it, then add and subtract.)} \\
&= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \\
&= \frac{3}{4}\left[1 + \left(\frac{4}{3}\right)\left(\frac{2x+1}{2}\right)^2\right] \\
&= \frac{3}{4}\left[1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2\right]
\end{aligned}$$

This enables us to compute the antiderivative family for the rational function $f(x) = \dfrac{3x^4 + 3x^2 - 1}{x^2 + x + 1}$. Indeed, observe

$$\begin{aligned}
\int \frac{3x^4 - 1}{x^2 + x + 1}dx &= \int \left(3x^2 - 3x + 3\right) dx - 4\int \frac{1}{x^2 + x + 1}dx \\
&= x^3 - \frac{3x^2}{2} + 3x - 4\int \frac{1}{(3/4)\left[1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2\right]}dx \\
&= x^3 - \frac{3x^2}{2} + 3x - \frac{16}{3}\int \frac{1}{1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2}dx \qquad \text{Let } u = \frac{2x+1}{\sqrt{3}} \\
&= x^3 - \frac{3x^2}{2} + 3x - \frac{8}{\sqrt{3}}\int \frac{1}{1 + u^2}du \\
&= x^3 - \frac{3x^2}{2} + 3x - \frac{8}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

Problem 1. Consider the improper rational function $f(x) = \dfrac{2x^4 - 4x^3 + 7x^2 + 6x - 14}{x^2 - 2x + 5}$.

Part a: Rewrite the function f as the sum of a polynomial and a proper rational function.

Part (b): Rewrite the denominator of the function f in the form $a(1 + u^2)$ for an appropriate constant a and linear function $u(x) = mx + b$.

Part (c): Find the antiderivative family for the function f.

- *Fundamental Theorem of Algebra:* It is possible to write any polynomial as the product of linear and irreducible quadratic factors. The linear factors correspond to the real roots of the polynomial, and the irreducible quadratic factors correspond to the (conjugate pairs) of complex roots for the polynomial.

There is a process by which we can decompose any *proper* rational function f into a sum of "partial" fractions, all of which can be integrated using the formulas mentioned at the beginning of this section. The denominators of the "partial" fractions come from the linear and irreducible quadratic factors for the denominator of f.

- The numerator of each "partial" fraction whose denominator is based on a linear factor will be a constant.
- The numerator of each "partial" fraction whose denominator is based on an irreducible quadratic factor will be a linear function.

Example 2 *Decompose the rational function* $f(x) = \dfrac{x}{2x^2+7x+3}$ *into a sum of "partial" fractions. Use this decomposition to compute the antiderivative family for* f.

Solution. First, note that $2x^2+7x+3 = (2x+1)(x+3)$. Therefore, it is possible to find constants A and B so that

$$\frac{x}{2x^2+7x+3} = \frac{A}{2x+1} + \frac{B}{x+3}$$

To find these constants, we first multiply both sides of the equation by the denominator on the left, cancel common factors on the right, then equate coefficients to create a system of equations. To see how this works, observe

$$\begin{aligned}
\frac{x}{2x^2+7x+3} = \frac{A}{2x+1} + \frac{B}{x+3} \quad &\Longrightarrow \quad x = (2x^2+7x+3)\left[\frac{A}{2x+1} + \frac{B}{x+3}\right] \\
&\Longrightarrow \quad x = (2x+1)(x+3)\left[\frac{A}{2x+1} + \frac{B}{x+3}\right] \\
&\Longrightarrow \quad x = A(x+3) + B(2x+1) \\
&\Longrightarrow \quad 1x + 0 = (A+2B)x + (3A+B)
\end{aligned}$$

Now, if we equate coefficients, we obtain the system of equations

$$\begin{cases} A + 2B = 1 \\ 3A + B = 0 \end{cases}$$

The second equations tells us that $B = -3A$. Substituting this into the first equation gives us

$$A - 6A = 1 \Longrightarrow A = -\frac{1}{5}$$

This information allows us to see that $B = 3/5$. Therefore, the "partial" fraction decomposition for the function f is given by

$$f(x) = \frac{x}{2x^2+7x+3} = \frac{3}{5(x+3)} - \frac{1}{5(2x+1)}$$

Now that we have the partial fraction decomposition for f, we can easily compute its antiderivative family.

$$\begin{aligned}
\int \frac{x}{2x^2+7x+3}dx &= \frac{3}{5}\int \frac{1}{x+3}dx - \frac{1}{5}\int \frac{1}{2x+1}dx \\
&= \frac{3}{5}\ln|x+3| - \frac{1}{10}\ln|2x+1| + C
\end{aligned}$$

****

Problem 2. Consider the rational function $f(x) = \dfrac{x+15}{x^2+2x-3}$.

Part (a): Construct the partial fraction decomposition for the function f.

Part (b): Use the partial fraction decomposition to compute the antiderivative family for the function f.

Example 3 *Decompose the rational function* $f(x) = \dfrac{4x^2+2x+1}{x^3+x}$ *into a sum of "partial" fractions. Use this decomposition to compute the antiderivative family for* f.

Solution. First, note that $x^3+x = x(x^2+1)$; therefore, the denominator of f has a linear and an irreducible quadratic factor. This tells us it is possible to find constants A, B, and C such that

$$\frac{4x^2+2x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

To find these constants, we follow the process used in the previous example. Observe

$$\begin{aligned}
\frac{4x^2+2x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad &\Longrightarrow \quad 4x^2+2x+1 = \left(x^3+x\right)\left[\frac{A}{x} + \frac{Bx+C}{x^2+1}\right] \\
&\Longrightarrow \quad 4x^2+2x+1 = A(x^2+1) + (Bx+C)(x) \\
&\Longrightarrow \quad 4x^2+2x+1 = (A+B)x^2 + Cx + A
\end{aligned}$$

Equating coefficients tells us that $A = 1$, $C = 2$, and $A + B = 4$. Of course, the last equation combined with the first tells us that $B = 3$. Therefore, we know

$$\frac{4x^2+2x+1}{x^3+x} = \frac{1}{x} + \frac{3x+2}{x^2+1}$$

We can use this decomposition to compute the antiderivative family for f. Observe

$$\begin{aligned}
\int \frac{4x^2+2x+1}{x^3+x}dx &= \int \frac{1}{x}dx + 3\int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx \\
&= \ln|x| + \frac{1}{2}\ln(1+x^2) + \arctan(x) + C
\end{aligned}$$

****

Problem 3. Consider the rational function $f(x) = \dfrac{4x^2+8x-6}{(x-4)(x^2+2)}$.

Part (a): Construct the "partial" fraction decomposition for the function f.

Part (b): Find the antiderivative family for the function f.

There is a wrinkle in the method of partial fraction decomposition that occurs when the denominator of the rational function has *repeated* factors. In this case, a "partial" fraction must be included for each power of the factor up to the power appearing in the factorization.

Example 4 *Find the antiderivative family for the rational function* $f(x) = \dfrac{x^2-5x+3}{x(x-1)^2}$.

Solution. We first must construct the "partial" fraction decomposition for f. In this case, the factor $x-1$ appears twice in the denominator. We handle this by introducing a "partial" fraction for $x-1$ and for $(x-1)^2$.

$$\begin{aligned}\frac{x^2-5x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \quad &\Longrightarrow \quad x^2-5x+3 = \big(x(x-1)^2\big)\left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right] \\ &\Longrightarrow \quad x^2-5x+3 = A(x-1)^2 + Bx(x-1) + Cx \\ &\Longrightarrow \quad x^2-5x+3 = (A+B)x^2 + (C-B-2A)x + A\end{aligned}$$

$$\left\{\begin{array}{rcrcrcr} A & + & B & & & = & 1 \\ -2A & - & B & + & C & = & -5 \\ A & & & & & = & 3 \end{array}\right.$$

Of course, the last equation tells us that $A = 3$. Substituting this value for A into the first equation tells us that $B = -2$. Substituting these two values into the second equation tells us that $C = -1$. Consequently, the "partial" fraction decomposition for f is

$$f(x) = \frac{x^2 - 5x + 3}{x(x-1)^2} = \frac{3}{x} - \frac{2}{x-1} - \frac{1}{(x-1)^2}$$

It is now possible to construct the antiderivative family for the function f. Observe

$$\begin{aligned} \int \frac{x^2 - 5x + 3}{x(x-1)^2} dx &= 3\int \frac{1}{x} dx - 2\int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^2} dx \\ &= 3\ln|x| - 2\ln|x-1| + \frac{1}{x-1} + C \end{aligned}$$

****

Problem 4. Consider the function $f(x) = \dfrac{8x^2 - 27x + 15}{(x+1)(2x-3)^2}$.

Part (a): Construct the "partial" fraction decomposition for the function f.

Part (b): Compute the antiderivative family for the function f.

HOMEWORK: Section 7.4 Page 501, Problems 7, 8, 9, 15, 16, 18, 19, 22, 23, 24, 29 (This one is tricky.)

- For Problem 29, observe the denominator is irreducible. Also, observe that $x + 4 = (x+1) + 3$ so

$$\int \frac{x+4}{x^2 + 2x + 5} dx = \int \frac{x+1}{x^2 + 2x + 5} dx + 3\int \frac{1}{x^2 + 2x + 5} dx$$