SERIES REPRESENTATIONS

PART I — GEOMETRIC SERIES

In these notes, we will introduce a way to approximate rational and transcendental functions using polynomials. The method developed represents one of the most powerful breakthroughs in nineteenth-century mathematics — one that still has widespread application today in biology, physics, and engineering. We begin with a definition.

Definition 1 *A geometric sum has the form*

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + ... + ar^n$$

where r and a are fixed constants. The constant r is called the common ratio of the geometric sum.

Geometric sums are of interest to scientists primarily because there is an elegant formula for computing them. This formula is the starting place for our exploration.

Theorem 2 *If r and a are fixed constants and n is a nonnegative integer, then as long as $r \neq 1$, we have*

$$\sum_{j=0}^{n} ar^j = \left(\frac{a}{1-r}\right)\left(1 - r^{n+1}\right)$$

Proof. To see why this formula is valid, first observe that

$$\begin{aligned}
(1-r)\sum_{j=0}^{n} ar^j &= (1-r)\left(a + ar + ar^2 + ... + ar^n\right) \\
&= (1-r)a + (1-r)ar + (1-r)ar^2 + ... + (1-r)ar^n \\
&= (a - ar) + (ar - ar^2) + (ar^2 - ar^3) + ... + (ar^n - ar^{n+1}) \\
&= a - ar^{n+1}
\end{aligned}$$

Consequently, we have

$$(1-r)\sum_{j=0}^{n} ar^j = a(1 - r^{n+1})$$

and the desired formula follows immediately.

QED

The key observation we take away from this formula is *not* that we have a closed form for a geometric sum — although this has uses in its own right. The most important insight coming from this formula is more subtle. This formula tells us that the *rational function*

$$f(r) = \frac{a}{1-r}\left(1 - r^{n+1}\right)$$

can be rewritten as a *polynomial*.

Example 3 *Rewrite the function* $f(x) = \dfrac{3}{1-x}\left(1 - x^5\right)$ *as a polynomial.*

Solution. Using Theorem 2 above, we see that

$$f(x) = 3 + 3x + 3x^2 + 3x^3 + 3x^4$$

****

It is worth noting that we could have developed this polynomial representation for the function f using long division, although it would have been tedious to do so. Once again, the real importance of the formula is more subtle. Consider the rational function

$$f(x) = \frac{a}{1-x}$$

This function is defined for all values of x except for $x = 1$. Now, suppose that we restrict the domain of f to the open segment $-1 < x < 1$. On this restricted domain, we know

$$\begin{aligned} f(x) &= \frac{a}{1-x} \\ &= \left(\frac{a}{1-x}\right)[1-0] \\ &= \left(\frac{a}{1-x}\right)\left[1 - \lim_{n \longrightarrow +\infty} x^{n+1}\right] \\ &= \lim_{n \longrightarrow +\infty} \left[\left(\frac{a}{1-x}\right)\left(1 - x^{n+1}\right)\right] \\ &= \lim_{n \longrightarrow +\infty} \sum_{j=0}^{n} ar^j \end{aligned}$$

The previous derivation tells us something crucial about the rational function f. As long as we restrict the domain of f to the open segment $-1 < x < 1$, polynomials of the form

$$G_n(x) = a + ax + ... + ax^n$$

will approximate the output of f. Furthermore, as the degree of these polynomials increases, the approximation improves.

Example 4 *Construct the first five approximating geometric polynomials for the function* $f(x) = \dfrac{4}{2+3x}$.

Solution. To begin, we rewrite the formula for f so that it has the proper form. Observe

$$f(x) = \frac{4}{2+3x} \Longrightarrow f(x) = \frac{4}{2-(-3x)} \Longrightarrow f(x) = \frac{2}{1-(-3x/2)}$$

Now, as long as we assume $-1 < \dfrac{3x}{2} < 1$, we know

$$f(x) = \lim_{n \longrightarrow +\infty} \sum_{j=0}^{n} 2\left(-\frac{3x}{2}\right)^j$$

It is now easy to construct the first five approximating geometric polynomials for the function f.

1. $G_0(x) = \sum_{j=0}^{0} 2\left(-\frac{3x}{2}\right)^j = 2$

2. $G_1(x) = \sum_{j=0}^{1} 2\left(-\frac{3x}{2}\right)^j = 2 - 3x$

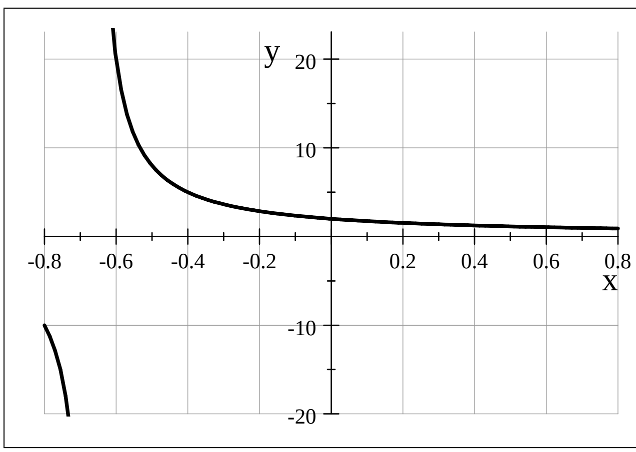
3. $G_2(x) = \sum_{j=0}^{2} 2\left(-\frac{3x}{2}\right)^j = 2 - 3x + \frac{9x^2}{2}$

4. $G_3(x) = \sum_{j=0}^{3} 2\left(-\frac{3x}{2}\right)^j = 2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4}$

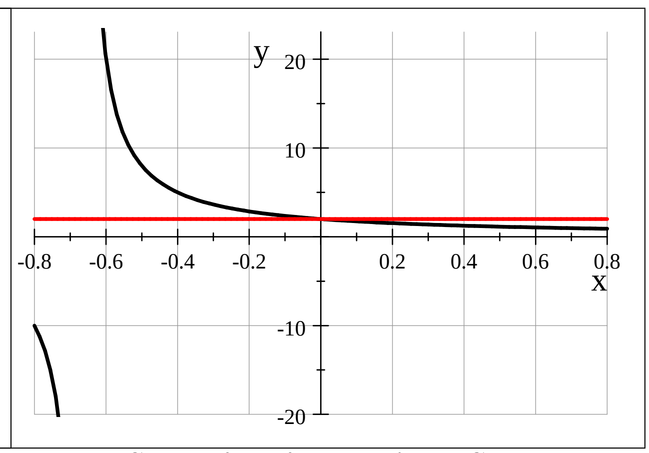
5. $G_4(x) = \sum_{j=0}^{4} 2\left(-\frac{3x}{2}\right)^j = 2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4} + \frac{81x^4}{8}$

****

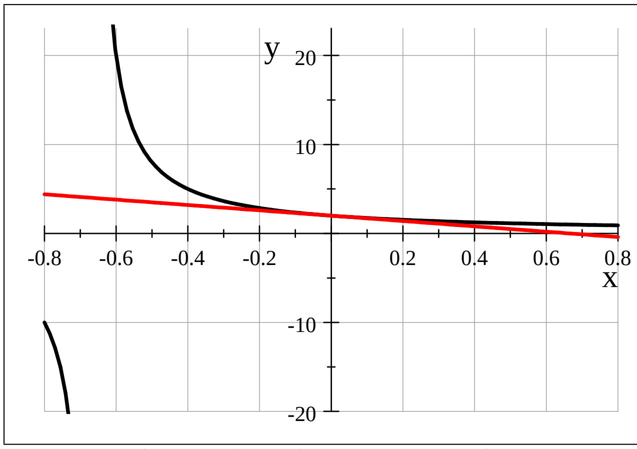
Notice how the approximating polynomials build on each other. Let's take a look at how well these polynomials do at approximating the graph of the function f on the open segment $-2/3 < x < 2/3$.



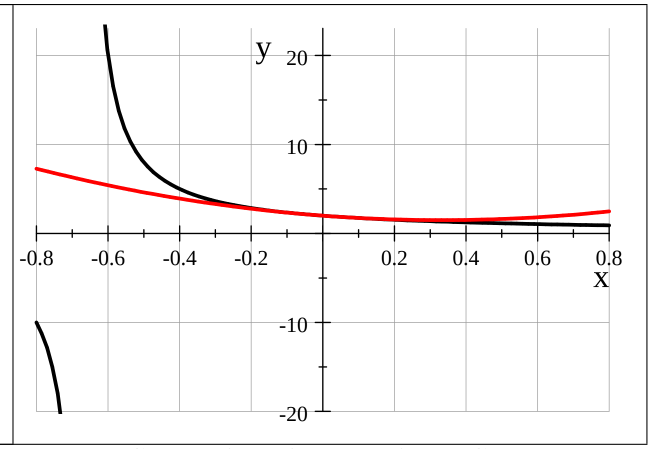
Graph of the function f



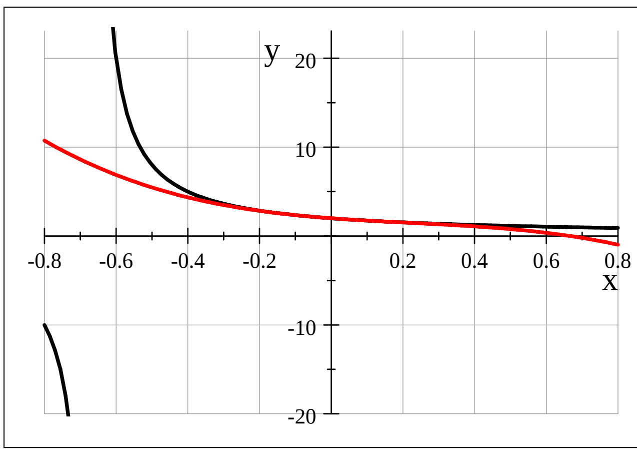
Graph of the function f and G_0



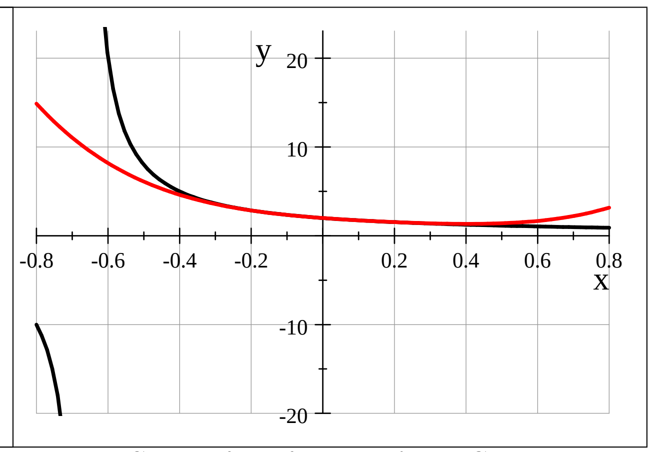
Graph of the function f and G_1



Graph of the function f and G_2



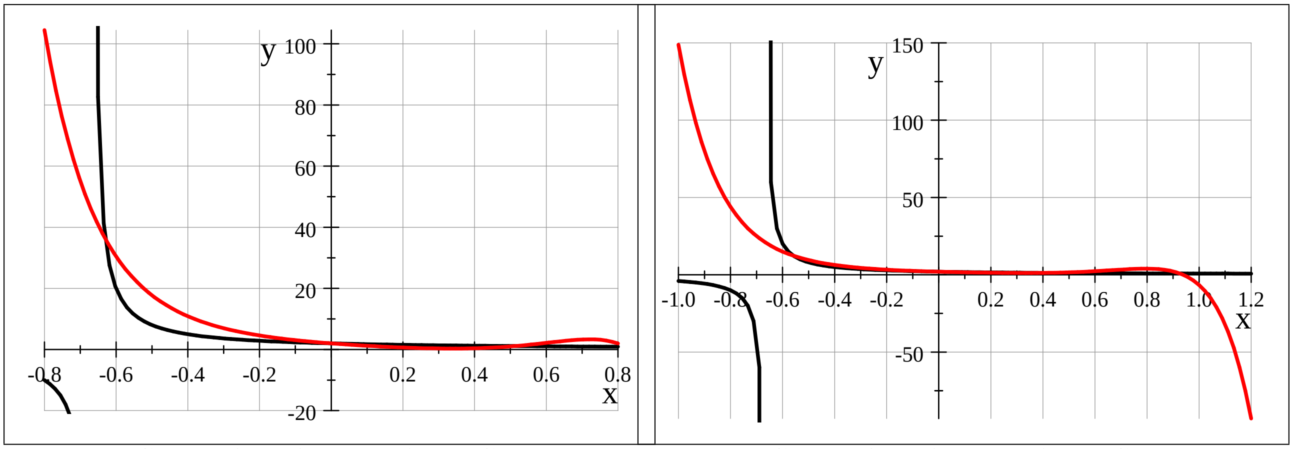
Graph of the function f and G_3



Graph of the function f and G_4

The graphs of the geometric polynomials are slowly providing better and better approximations to the graph of the function f on the open $-2/3 < x < 2/3$. However, outside of this open segment, all bets are off. This becomes especially apparent if we consider a higher-degree approximating polynomial, say

$$G_9(x) = 2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4} + \frac{81x^4}{8} - \frac{243x^5}{16} + \frac{729x^6}{32} - \frac{2187x^7}{64} + \frac{6561x^8}{128} - \frac{19683x^9}{256}$$



Graph of the function f and G_9 | Graph of the function f and G_9

Problem 1. Construct the first six geometric polynomials that approximate the function $f(x) = \dfrac{1}{7+4x}$. On what open segment do the polynomials provide increasingly good approximations to the function f?

Definition 5 *Let m and n be nonnegative integers with $m \leq n$. A geometric series is a limit of the form*

$$\sum_{j=m}^{\infty} ar^j = \lim_{n \longrightarrow +\infty} \left(ar^m + ar^{m+1} + ... + ar^n \right)$$

We say the geometric series coverges if the limit is finite, and say it diverges otherwise.

Example 6 *Determine the value of the series* $\displaystyle\sum_{j=10}^{\infty} (-1)^j \left(\frac{2^{j+1}}{3^j} \right)$.

Solution. The key to determining the value of this series is to rewrite it in the form

$$\sum_{j=0}^{\infty} ar^j$$

for an appropriate constant a and common ratio r.

$$
\begin{aligned}
\sum_{j=10}^{\infty} (-1)^j \left(\frac{2^{j+1}}{3^j}\right) &= \sum_{j=10}^{\infty} 2 \cdot \left(-\frac{2}{3}\right)^j \\
&= \sum_{j=0}^{\infty} 2 \cdot \left(-\frac{2}{3}\right)^j - \sum_{j=0}^{9} 2 \cdot \left(-\frac{2}{3}\right)^j \\
&= \left(\frac{2}{1+2/3}\right) - \left(\frac{2}{1+2/3}\right)\left(1-(-2/3)^{10}\right) \\
&= \left(\frac{2}{1+2/3}\right)\left[1 - 1 + \left(\frac{2}{3}\right)^{10}\right] \\
&= \left(\frac{3}{5}\right)\left(\frac{2^{11}}{3^{10}}\right) \\
&= \frac{2040}{98415}
\end{aligned}
$$

****

Problem 2. Compute the value of the following series.

(a) $\displaystyle\sum_{j=3}^{\infty} \frac{2^{j+2}}{3^{j+1}}$

(b) $\displaystyle\sum_{j=5}^{\infty} (-1)^j \frac{3^{j-1}}{5^j}$

Problem 3. Consider the repeating decimal 0.134134134....

Part (a): Explain why $0.134134134... = \displaystyle\sum_{j=1}^{\infty} 134 \left(\frac{1}{1000}\right)^j$.

Part(b): Use Part (a) and the formula for a convergent geometric series to rewrite 0.134134134... as the ratio of two integers.

Example 7 *Show that the series* $\sum_{j=0}^{\infty}(-1)^j\left(\frac{3}{2}\right)^{j+2}$ *diverges.*

Solution. First, observe that

$$\sum_{j=0}^{\infty}(-1)^j\left(\frac{3}{2}\right)^{j+2}=\sum_{j=0}^{\infty}9\cdot\left(-\frac{3}{2}\right)^j$$

Now, for any positive integer n, consider the geometric sum

$$\begin{aligned}\sum_{j=0}^{n}9\cdot\left(-\frac{3}{2}\right)^j &= \left(\frac{9}{1+3/2}\right)\left(1-\left(-\frac{3}{2}\right)^{n+1}\right)\\ &= \left(\frac{18}{5}\right)\left(1-\left(-\frac{3}{2}\right)^{n+1}\right)\end{aligned}$$

The value of this geometric sum oscillates wildly between ever more positive and ever more negative values as n increases. For example,

- When $n=1$ we have $\sum_{j=0}^{1}9\cdot\left(-\frac{3}{2}\right)^j=-\frac{9}{10}$.
- When $n=4$ we have $\sum_{j=0}^{4}9\cdot\left(-\frac{3}{2}\right)^j=\frac{495}{16}$.
- When $n=9$ we have $\sum_{j=0}^{9}9\cdot\left(-\frac{3}{2}\right)^j=-\frac{104\,445}{512}$.

Now, since we know

$$\sum_{j=0}^{\infty}(-1)^j\left(\frac{3}{2}\right)^{j+2}=\lim_{n\longrightarrow+\infty}\sum_{j=0}^{n}9\cdot\left(-\frac{3}{2}\right)^j$$

we also know that this limit process does not "settle down" to any fixed value. Consequently, the limit does not exist; and we must conclude that the series in question diverges.

****

Theorem 8 *A geometric series converges if and only if its common ratio* r *is strictly between* -1 *and* 1*.*

Proof. If $-1<r<1$, then we already know that the geometric series converges. Indeed, we know

$$\begin{aligned}\sum_{j=m}^{\infty}ar^j &= \sum_{j=0}^{\infty}ar^j-\sum_{j=0}^{m-1}ar^j\\ &= \frac{a}{1-r}-\left(\frac{a}{1-r}\right)(1-r^m)\\ &= \frac{ar^m}{1-r}\end{aligned}$$

On the other hand, suppose that $1\leq r$. This tells us that $r^m+r^{m+1}+...+r^n\geq(1)^m+...+(1)^n=(n-m)$. Consequently,

$$\sum_{j=m}^{\infty}ar^j=\lim_{n\longrightarrow+\infty}\left(ar^m+ar^{m+1}+...+ar^n\right)\geq\lim_{n\longrightarrow+\infty}(n-m)\,a=+\infty\ \text{ if } a>0$$

$$\sum_{j=m}^{\infty} ar^j = \lim_{n \longrightarrow +\infty} \left(ar^m + ar^{m+1} + ... + ar^n \right) \leq \lim_{n \longrightarrow +\infty} (n-m)\, a = -\infty \quad \text{if } a < 0$$

This tells us that the series diverges.

Finally, suppose that $r \leq -1$. In this case, for each positive integer n, we know

$$r^n = (-|r|)^n = (-1)^n \cdot |r|^n$$

Since $|r| \geq 1$, we know that $|r|^n$ grows larger and larger as n increases. Consequently, r^n alternates between being more and more positive and more and more negative (as the value of n alternates between being even and odd). As a result, the value of the geometric sums

$$\sum_{j=0}^{n} ar^j$$

will oscillate between ever more negative and ever more positive numbers as n increases. We must therefore conclude that the geometric series diverges, since the limit of these geometric sums does not exist.

Example 9 *Construct a series representation for the function* $f(x) = \dfrac{1}{x}$.

Solution. First, observe that

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)}$$

Therefore, if we restrict the domain of f to the open segment $-1 < 1-x < 1$ (that is, restrict the domain to the set $0 < x < 2$) we know

$$f(x) = \sum_{j=0}^{\infty} (1-x)^j$$

The open segment $0 < x < 2$ is called the *interval of convergence* for this series representation for f. The series representation is not valid outside of this interval.

****

Example 10 *Construct two different series representations for the function* $f(x) = \dfrac{1}{5-4x}$.

Solution. First, observe that

$$f(x) = \frac{1}{5-4x} = \frac{(1/5)}{1-(4x/5)} = \sum_{j=0}^{\infty} \left(\frac{1}{5}\right)\left(\frac{3x}{2}\right)^j$$

This series representation will only be valid as long as we restrict the domain of f to the open segment $-1 < 3x/2 < 1$ (that is, the interval of convergence for this series representation is the set $-2/3 < x < 2/3$).

Next, observe that

$$f(x) = \frac{1}{5-4x} = \frac{1}{1-4(x-1)} = \sum_{j=0}^{\infty} 4^j (x-1)^j$$

This series representation will only be valid as long as we restrict the domain of f to the open segment $-1 < 4(x-1) < 1$ (that is, the interval of convergence for this series representation is the set $3/4 < x < 5/4$).

****

Problem 4. Construct two different series representations for $f(x) = \dfrac{2}{5+8x}$. What is the interval of convergence for each representation?

Problem 5. Consider the function $f(x) = \dfrac{1}{1+x^2}$. If we restrict the domain of this function to the interval $-1 < x < 1$, explain why we have the following series representation.

$$f(x) = \sum_{j=0}^{\infty} (-1)^j x^{2j}$$

HOMEWORK FOR PART I.

Determine the value of each of the following convergent series.

(1) $\displaystyle\sum_{j=0}^{\infty} (-1)^j \left(\frac{3}{4}\right)^j$ (2) $\displaystyle\sum_{j=0}^{\infty} \frac{1-2^j}{3^j}$ (3) $\displaystyle\sum_{j=3}^{\infty} \frac{4^{j+1}}{5^j}$

(4) $\displaystyle\sum_{j=2}^{\infty} \frac{4^{2j}}{5^{2j-1}}$ (5) $\displaystyle\sum_{j=4}^{\infty} (-1)^j \frac{3-4^j}{6^{2j}}$ (6) $\displaystyle\sum_{j=0}^{\infty} (-1)^j \frac{2-2^{j+1}}{7^j}$

7. Construct two series representations for the function $f(x) = \dfrac{3}{4-5x}$. What is the interval of convergence for each representation?

8. Construct a series representation for the function $f(x) = \dfrac{2}{1-x^2}$. What is the interval of convergence for this representation?

9. Construct a series representation for the function $f(x) = \dfrac{1}{1+8x^3}$. What is the interval of convergence for this representation?

10. Find the rational number that is equal to the repeating decimal 0.121212....

11. Use a geometric series to show that the repeating decimal 0.9999... is equal to the repeating decimal 1.000....

PART II — TAYLOR POLYNOMIALS

Constructing series representations for simple rational functions might be an interesting exercise, but the process has little practical significance. It is easy to evaluate simple rational functions directly, so there is not much reason to approximate them using polynomials. It is another story with *transcendental* functions, however. There is no straightforward way to determine the output for a given transcendental function associated with most inputs.

Let's consider an example. The function $f(x) = \arctan(x)$ is transcendental — there is no algebraic formula that tells us how to obtain the output of this function from its input values. However, there is a peculiar coincidence associated with this function.

- We know that $\displaystyle\int_0^x \frac{1}{1+t^2}\,dt = \arctan(t)$.
- We know that, as long as we restrict $-1 < t < 1$, we have $\displaystyle\frac{1}{1+t^2} = \sum_{j=0}^{\infty}(-1)^j t^{2j}$.

Now, for any positive integer n, we know

$$\frac{1}{1+t^2} \approx 1 - t^2 + t^4 - t^6 + ... + (-1)^n t^{2n}$$

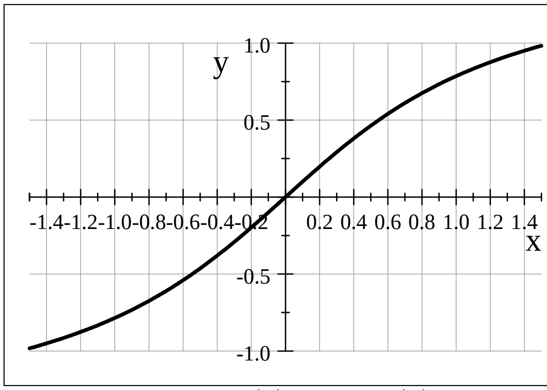
and we know that this approximation gets better and better as n gets larger. It stands to reason that

$$\begin{aligned}
\arctan(x) &= \int_0^x \frac{1}{1+t^2}\,dt \\
&\approx \int_0^x \left[1 - t^2 + t^4 - t^6 + ... + (-1)^n t^{2n}\right] dt \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + (-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

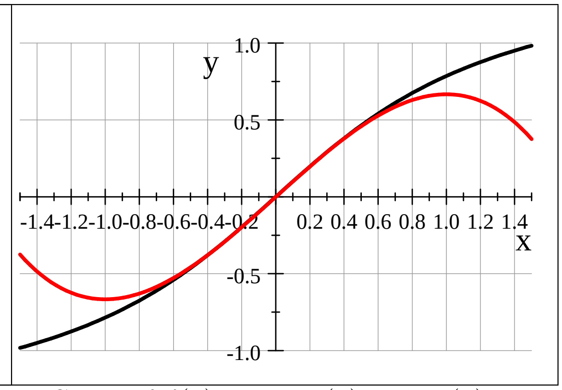
It also stands to reason that this approximation *ought* to get better and better as n gets larger; however, we won't address that question at the moment.

Is it really the case that the polynomials we created above give better and better approximations to the arctangent function on the open segment $-1 < x < 1$? Let's check a few and see. Let

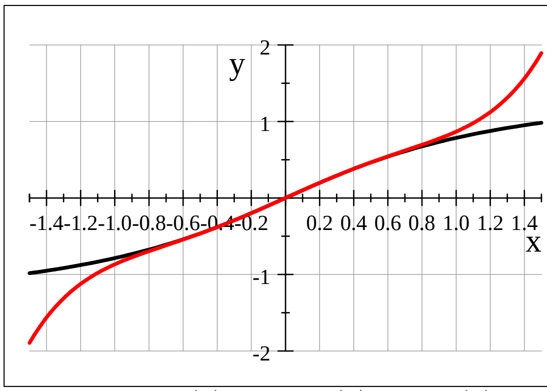
$$P_n(x) = \sum_{j=0}^{n}(-1)^j \frac{x^{2j+1}}{2j+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + (-1)^n \frac{x^{2n+1}}{2n+1}$$



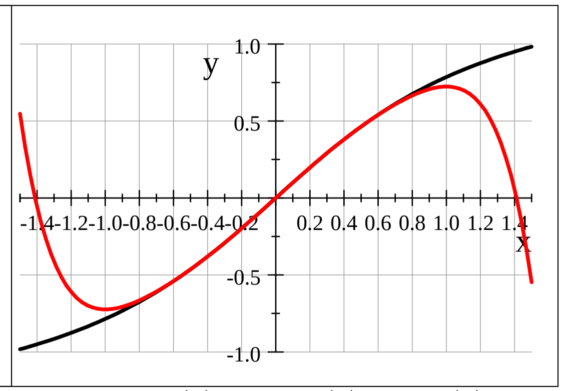
Graph of $f(x) = \arctan(x)$



Graph of $f(x) = \arctan(x)$ and $P_1(x)$



Graph of $f(x) = \arctan(x)$ and $P_2(x)$



Graph of $f(x) = \arctan(x)$ and $P_3(x)$

The graphs above certainly lend credence to the idea that the polynomials P_n give better and better approximations to the graph of the arctangent function on the open segment $-1 < x < 1$. Outside of this open segment, there appears to be no correlation between the polynomial graphs and the graph of the arctangent.

Problem 1. Here is another strange coincidence.

- We know that, as long as $x > 0$, we have $\displaystyle\int_1^x \frac{1}{t}\,dt = \ln(x)$.
- We know that, as long as we restrict $0 < t < 2$, we have $\displaystyle\frac{1}{t} = \sum_{j=0}^{\infty}(1-t)^j = \sum_{j=0}^{\infty}(-1)^j(t-1)^j$.

Part (a): Use these observations to construct approximating polynomials $P_n(x)$ for the natural logarithm function.

Part (b): Use your graphing calculator to compare a few of these approximating polynomials to the graph of the natural logarithm function. On what interval do these polynomials approximate the function?

It seems possible to devise families of polynomials that provide increasingly good approximations to *some* transcendental functions — at least those whose derivatives are simple rational functions. However, there is a way to find approximating polynomials for virtually any transcendental function. To see how, let's consider an arbitrary function g that is infinitely differentiable (possesses derivatives of all orders) at some fixed input value $x = a$. For simplicity, for any nonnegative integer j, we will adopt the notation

$$g^{(j)}(x) = \begin{cases} g(x) & \text{if} \quad j = 0 \\ \dfrac{d^j}{dx^j}\,[g(x)] & \text{if} \quad j > 0 \end{cases}$$

- We already know a very simple first-degree polynomial that approximates the graph of the function g near the point $(a, g(a))$ — namely the tangent line $T_1(x) = g(a) + g^{(1)}(a)\,[x - a]$.

The tangent line to the graph of g at $(a, g(a))$ matches the increasing/decreasing behavior of the graph of g at the point of tangency. If we want to create a polynomial that fits the graph of g better, we should engineer this polynomial so that it also matches the *concavity* of g at the point of tangency. In other words, we want to engineer a polynomial $T_2(x)$ so that the following conditions are met

1. $T_2(a) = g(a)$
2. $T_2^{(1)}(a) = g^{(1)}(a)$
3. $T_2^{(2)}(a) = g^{(2)}(a)$

Consider the polynomial

$$T_2(x) = g(a) + g^{(1)}(a)\,[x - a] + c\,[x - a]^2$$

where c is an undetermined coefficient. Notice that T_2 is constructed to that it automatically satisfies Conditions 1 and 2. Indeed,

$$T_2(a) = g(a) + g^{(1)}(a)\,[a - a] + c\,[a - a]^2 = g(a)$$

$$\begin{aligned} T_2^{(1)}(x) &= \frac{d}{dx}\left[g(a) + g^{(1)}(a)\,[x - a] + c\,[x - a]^2\right] \\ &= g^{(1)}(a) + 2c\,[x - a] \end{aligned}$$

The last equation tells us that $T_2^{(1)}(a) = g^{(1)}(a)$. The question we need to answer is simply, "what should be the value of c?" We *want* to have $T_2^{(2)}(a) = g^{(2)}(a)$, so lets assume this is true and see what that assumption tells us about the value of c. Now,

$$T_2^{(1)}(x) = \frac{d}{dx}\left[g^{(1)}(a) + 2c\,[x - a]\right] = 2c$$

Consequently, if we want to have $T_2^{(2)}(a) = g^{(2)}(a)$, it follows that we must let $c = g^{(2)}(a)/2$. In other words, we can let

$$T_2(x) = g(a) + g^{(1)}(a)\,[x-a] + \frac{g^{(2)}(a)}{2}\,[x-a]^2$$

Now, suppose that we want to construct a polynomial $T_3(x)$ that is an even better match to the graph of g near the input value $x = a$. Not only should T_3 match the increase/decrease and concavity of the graph of g at $x = a$, it stands to reason that T_3 should also match the *rate of change* for the concavity of g at $x = a$. In other words, if the graph of g has only a slight upward bend at $x = a$, then so should the graph of T_3. Or, if the graph of g has a very sharp bend downward at $x = a$, then so should the graph of T_3. We want to engineer the polynomial T_3 so that

1. $T_3(a) = g(a)$
2. $T_3^{(1)}(a) = g^{(1)}(a)$
3. $T_3^{(2)}(a) = g^{(2)}(a)$
4. $T_3^{(3)}(a) = g^{(3)}(a)$

Consider the polynomial $T_3(x) = g(a) + g^{(1)}(a)\,[x-a] + \dfrac{g^{(2)}(a)}{2}\,[x-a]^2 + c\,[x-a]^3$.

The polynomial T_3 has been engineered to automatically satisfy Conditions 1 – 3. Observe that

$$\begin{aligned}
T_3^{(3)}(x) &= \frac{d^3}{dx^3}\left[g(a) + g^{(1)}(a)\,[x-a] + \frac{g^{(2)}(a)}{2}\,[x-a]^2 + c\,[x-a]^3\right] \\
&= \frac{d^2}{dx^2}\left[g^{(1)}(a) + g^{(2)}(a)[x-a] + 3c[x-a]^2\right] \\
&= \frac{d}{dx}\left[g^{(2)}(a) + (3\cdot 2)c[x-a]\right] \\
&= (3\cdot 2)c
\end{aligned}$$

If we want to have $T_3^{(3)}(a) = g^{(3)}(a)$, then it is clear that we must let $c = \dfrac{g^{(3)}(a)}{3\cdot 2}$. Consequently, the polynomial we seek is

$$T_3(x) = g(a) + \frac{g^{(1)}(a)\,[x-a]}{1} + \frac{g^{(2)}(a)}{2\cdot 1}\,[x-a]^2 + \frac{g^{(3)}(a)}{3\cdot 2\cdot 1}\,[x-a]^3$$

There is a clear pattern emerging as we engineer these polynomials. In mathematics, it is customary to let $n!$ represent the product of the first n positive integers. We will adopt this convention in the following definition.

Definition 11 *Suppose that g is an infinitely differentiable function at the input value $x = a$. For a fixed positive integer n, the nth degree Taylor polynomial for g at $x = a$ is defined by*

$$\begin{aligned}
T_n(x,a) &= g(a) + \sum_{j=1}^{n} \frac{g^{(j)}(a)}{j!}\,[x-a]^j \\
&= g(a) + \frac{g^{(1)}(a)}{1}\,[x-a] + \frac{g^{(2)}(a)}{2\cdot 1}\,[x-a]^2 + \frac{g^{(3)}(a)}{3\cdot 2\cdot 1}\,[x-a]^3 + \ldots + \frac{g^{(n)}(a)}{n\cdot(n-1)\cdot\ldots\cdot 3\cdot 2\cdot 1}[x-a]^n
\end{aligned}$$

The nth degree Taylor polynomial $T_n(x,a)$ for a function g has been engineered so that $T_n(a,a) = g(a)$ and $T_n^{(j)}(a,a) = g^{(j)}(a)$ for $1 \le j \le n$. Therefore, these polynomials *should* provide increasingly good approximations to the graph of g near the input value $x = a$.

Example 12 *Construct the Taylor polynomials* $T_1(x,0)$*,* $T_2(x,0)$*,* $T_3(x,0)$*, and* $T_4(x,0)$ *for* $f(x) = \arctan(x)$*.*

Solution. We will need the first four derivatives of the arctangent function. While tedious, computation of these derivatives is straightforward.

$$f^{(1)}(x) = \frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2} \qquad f^{(2)}(x) = \frac{d^2}{dx^2}\left[\arctan(x)\right] = -\frac{2x}{(1+x^2)^2}$$

$$f^{(3)}(x) = \frac{d^3}{dx^3}\left[\arctan(x)\right] = \frac{2(3x^2-1)}{(1+x^2)^3} \qquad f^{(4)}(x) = \frac{d^4}{dx^4}\left[\arctan(x)\right] = -\frac{24x(x^2-1)}{(1+x^2)^4}$$

With the derivative formulas in hand, we can now construct the Taylor polynomials.

$$\begin{aligned} T_1(x,0) &= f(0) + \frac{f^{(1)}(0)}{1}\left[x-0\right] \\ &= 0 + \frac{x}{1} \\ &= x \end{aligned}$$

$$\begin{aligned} T_2(x,0) &= f(0) + \frac{f^{(1)}(0)}{1}\left[x-0\right] + \frac{f^{(2)}(0)}{2\cdot 1}\left[x-0\right]^2 \\ &= \frac{x}{1} + \frac{0}{2\cdot 1}\cdot\left[x-0\right]^2 \\ &= x \end{aligned}$$

$$\begin{aligned} T_3(x,0) &= f(0) + \frac{f^{(1)}(0)}{1}\left[x-0\right] + \frac{f^{(2)}(0)}{2\cdot 1}\left[x-0\right]^2 + \frac{f^{(3)}(0)}{3\cdot 2\cdot 1}\left[x-0\right]^3 \\ &= x + 0\cdot\left[x-0\right]^2 - \left(\frac{2}{3\cdot 2\cdot 1}\right)\left[x-0\right]^3 \\ &= x - \frac{x^3}{3} \end{aligned}$$

$$\begin{aligned} T_4(x,0) &= f(0) + \frac{f^{(1)}(0)}{1}\left[x-0\right] + \frac{f^{(2)}(0)}{2\cdot 1}\left[x-0\right]^2 + \frac{f^{(3)}(0)}{3\cdot 2\cdot 1}\left[x-0\right]^3 + \frac{f^{(4)}(0)}{3\cdot 2\cdot 1}\left[x-0\right]^4 \\ &= \frac{x}{1} + \frac{0}{2\cdot 1}\cdot\left[x-0\right]^2 - \left(\frac{2}{3\cdot 2\cdot 1}\right)\left[x-0\right]^3 + \left(\frac{0}{4\cdot 3\cdot 2\cdot 1}\right)\left[x-0\right]^4 \\ &= x - \frac{x^3}{3} \end{aligned}$$

****

Notice that the Taylor polynomials we are creating come in pairs — $T_{2n}(x,0) = T_{2n+1}(x,0)$. Notice also that the Taylor polynomials we are creating are the same as the polynomials $P_n(x)$ we created for the arctangent function by termwise integration of the series representation for $(1+x^2)^{-1}$.

Problem 2. Create the Taylor polynomials $T_1(x,0)$, $T_2(x,0)$, $T_3(x,0)$, and $T_4(x,0)$ for $f(x) = \dfrac{4}{2+3x}$. Compare these to the polynomials $G_1(x)$, ... $G_4(x)$ appearing in Example 4 of Part I.

$$f^{(1)}(x) = -\frac{12}{(2+3x)^2} \qquad f^{(2)}(x) = \frac{72}{(2+3x)^3} \qquad f^{(3)}(x) = -\frac{648}{(2+3x)^4} \qquad f^{(4)}(x) = \frac{7776}{(2+3x)^5}$$

Problem 3. Create the Taylor polynomials $T_1(x,1)$, $T_2(x,1)$, $T_3(x,1)$, and $T_4(x,1)$ for $f(x) = \arctan(x)$. (Remember that $\arctan(1) = \dfrac{\pi}{4}$.)

The Taylor polynomials $T_1(x,1)$, $T_2(x,1)$, $T_3(x,1)$, and $T_4(x,1)$ for $f(x) = \arctan(x)$ are

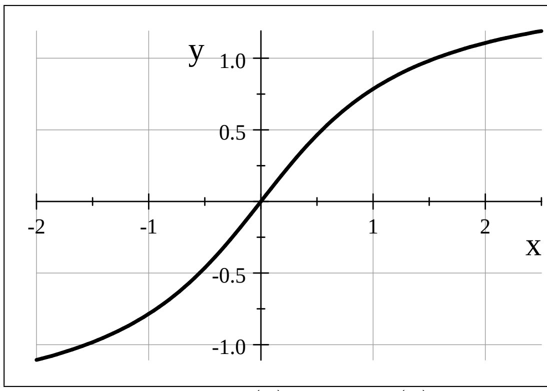
1. $T_1(x,1) = \dfrac{\pi}{4} + \dfrac{x-1}{2}$

2. $T_2(x,1) = \dfrac{\pi}{4} + \dfrac{x-1}{2} - \dfrac{(x-1)^2}{4}$

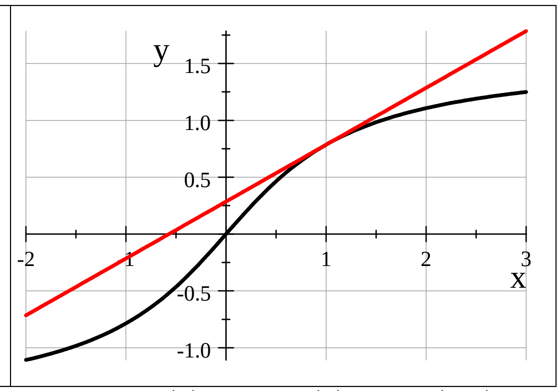
3. $T_3(x,1) = \dfrac{\pi}{4} + \dfrac{x-1}{2} - \dfrac{(x-1)^2}{4} + \dfrac{(x-1)^3}{12}$

4. $T_4(x,1) = \dfrac{\pi}{4} + \dfrac{x-1}{2} - \dfrac{(x-1)^2}{4} + \dfrac{(x-1)^3}{12}$

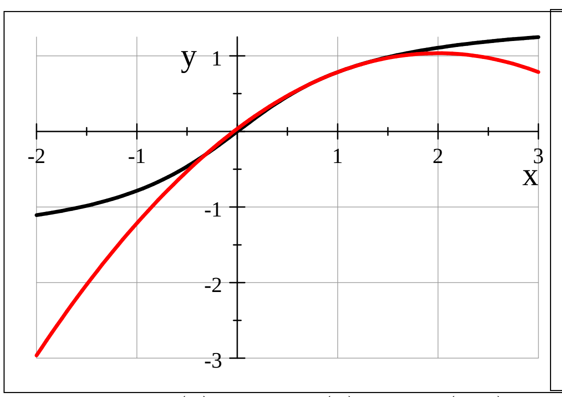
Notice that T_3 and T_4 happen to be the same this time. Let's compare the graphs of these polynomials to the graph of the arctangent function.



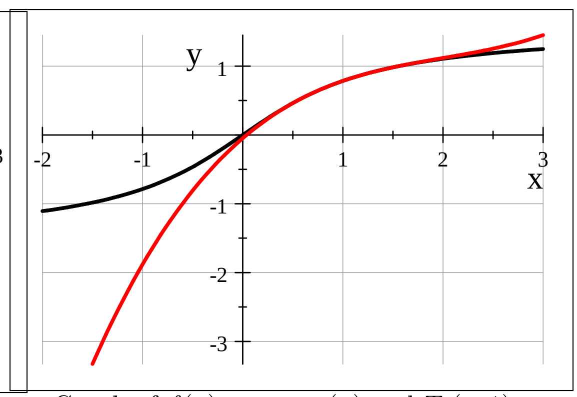
Graph of $f(x) = \arctan(x)$



Graph of $f(x) = \arctan(x)$ and $T_1(x,1)$



Graph of $f(x) = \arctan(x)$ and $T_2(x,1)$



Graph of $f(x) = \arctan(x)$ and $T_3(x,1)$

In this case, it appears that the interval of convergence has shifted compared to the interval of convergence we obtained when we constructed the polynomials P_n for the arctangent at the start of this section. The Taylor polynomials $T_n(x,1)$ for the arctangent appears to have $0 < x < 2$ as its interval of convergence — although we cannot yet be certain of this.

HOMEWORK FOR PART II

1. Construct the Taylor polynomials $T_1(x,1)$ through $T_4(x,1)$ for the function $f(x) = \sqrt{x}$.

2. Construct the Taylor polynomials $T_1(x,0)$ through $T_5(x,0)$ for the function $f(x) = e^x$. How well does $T_3(1,0)$ approximate the value of e? How well does $T_5(1,0)$ approximate the value of e?

3. Based on your work in Problem 1, construct a formula for $T_n(x,0)$ for $f(x) = e^x$, where n is any positive integer.

4. Construct the Taylor polynomials $T_1(x,0)$ through $T_6(x,0)$ for the function $f(x) = \sin(x)$.

5. Construct the Taylor polynomials $T_1(x,0)$ through $T_6(x,0)$ for the function $f(x) = \cos(x)$.

6. Construct the Taylor polynomials $T_1(x,\pi/4)$ through $T_4(x,\pi/4)$ for the function $f(x) = \cos(x)$. Remember, $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

7. Construct the Taylor polynomials $T_1(x,\pi/4)$ through $T_4(x,\pi/4)$ for the function $f(x) = \tan(x)$. How well does $T_4(\pi/3,\pi/4)$ approximate the value of $\tan(\pi/3)$? How well does $T_4(0,\pi/4)$ approximate the value of $\tan(0)$?

8. Construct the Taylor polynomials $T_1(x,0)$ through $T_5(x,0)$ for the function $f(x) = \sin(x)\cos(x)$.

9. Construct theTaylor polynomial $T_3(x,1/2)$ for the function $f(x) = \arcsin(x)$. Remember, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

10. Construct the Taylor polynomial $T_6(x,1)$ for the function $f(x) = x \ln x$.

PART III — INTERVALS OF CONVERGENCE

In this section, we will explore whether or not Taylor polynomials created for a function f actually converge to the function on some interval. Answering this question fully requires extensive technical machinery that is beyond the scope of this course, but we can give partial answers. We begin with a definition.

Definition 13 *A series is a special limit of the form*

$$\sum_{j=p}^{\infty} f_j(x) = \lim_{n \longrightarrow +\infty} \sum_{j=p}^{n} f_j(x)$$

where each f_j is a function of x that indexed by j, and p is a nonnegative integer (usually 0 or 1).

A series is the limit of a sequence of finite sums; these finite sums are often called the *partial sums* of the series. Infinite series are sometimes called *infinite sums*, although this terminology doesn't really make sense. We say that a series *converges* if the limit of its partial sums exists and is finite. It is traditional to call the value of this limit the *sum* of the series. We say the series *diverges* provided the limit of the partial sums fails to exist.

Whether a series converges or diverges can be very sensitive to small changes in the functions that comprise the partial sums. For example, consider the series

$$\sum_{j=1}^{\infty} \frac{1}{j} \qquad \text{and} \qquad \sum_{j=0}^{\infty} (-1)^j \frac{1}{j+1}$$

Both of these series have their partial sums made up of constant functions. In the first series, we have $f_j(x) = j^{-1}$, and in the second series, we have $f_j(x) = (-1)^j \cdot j^{-1}$.

The first series is called the *harmonic series*, and it diverges. To see why, consider a pattern that arises in the partial sums.

- $\displaystyle\sum_{j=1}^{2} \frac{1}{j} = 1 + \frac{1}{2} > 2 \cdot \frac{1}{2}$

- $\displaystyle\sum_{j=1}^{4} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} > 3 \cdot \frac{1}{2}$

- $\displaystyle\sum_{j=1}^{8} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > 4 \cdot \frac{1}{2}$

Proceeding in this manner, it can be proven that for any positive integer m, we have

$$\sum_{j=1}^{2^m} \frac{1}{j} > (m+1) \cdot \frac{1}{2}$$

Now, for any positive integer $n > 1$, there exists a largest positive integer m such that $2^m \leq n$. Consequently, we know that

$$\sum_{j=1}^{n} \frac{1}{j} \geq \sum_{j=1}^{2^m} \frac{1}{j} > (m+1) \cdot \frac{1}{2}$$

Therefore, we are forced to conclude that

$$\lim_{n \longrightarrow +\infty} \sum_{j=1}^{n} \frac{1}{j} \geq \lim_{m \longrightarrow +\infty} (m+1) \cdot \frac{1}{2} = +\infty$$

The second series is called the *alternating* harmonic series. It turns out that this series *converges.* Formally proving this fact is beyond the scope of this course, but we can give some motivation for the claim. Using a computer algebra system, it is a routine matter to determine

- $\displaystyle\sum_{j=0}^{2} (-1)^j \frac{1}{j+1} = 1 - \frac{1}{2} + \frac{1}{3} \approx 0.833\,33$

- $\displaystyle\sum_{j=0}^{10} (-1)^j \frac{1}{j+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \approx 0.645\,63$

- $\displaystyle\sum_{j=0}^{100}(-1)^j\frac{1}{j+1}\approx 0.698\,07$

- $\displaystyle\sum_{j=0}^{1000}(-1)^j\frac{1}{j+1}\approx 0.693\,65$

- $\displaystyle\sum_{j=0}^{10000}(-1)^j\frac{1}{j+1}\approx 0.693\,20$

It appears that the values of the partial sums are slowly converging on a fixed number, which is approximately 0.693.

Definition 14 *A power series is a series of the form*

$$\sum_{j=p}^{\infty} A_j\,(x-a)^{mj+b}$$

where the coefficients A_j are real numbers, and $p, m,$ and b are fixed integers with p nonnegative. We say the series is centered on the number $x = a$.

The partial sums of a power series are polynomials.

$$P_n(x) = \sum_{j=p}^{n} A_j\,(x-a)^{mj+b}$$

Every geometric series is a special example of a power series. By taking the limit of Taylor polynomials we create for any function, we also form a power series. Not surprisingly, this power series is called a *Taylor series* for the function.

Definition 15 *Let f be an infinitely differentiable function for an input value $x = a$. The Taylor series for f centered at $x = a$ is given by*

$$f(a) + \sum_{j=1}^{\infty}\left(\frac{f^{(j)}(a)}{j!}\right)[x-a]^j$$

Theorem 16 *Suppose that $\displaystyle\sum_{j=p}^{\infty} A_j\,(x-a)^{mj+b}$ and suppose that all the coefficients A_j are nonzero. Exactly one of the following statements will always be true.*

1. *The series will only converge when $x = a$.*
2. *The series will converge for every value of x.*
3. *There exists a positive number R such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.*

The key to this theorem lies in the following computation:

$$\text{For any power series with all } A_j \neq 0, \text{ let } R = \lim_{j\longrightarrow+\infty}\left|\frac{A_j}{A_{j+1}}\right|$$

- If $R = 0$, then the power series will only converge when $x = a$.
- If $R = +\infty$, then the power series will converge for every value of x.
- If R is positive, then the series will converge for $|x - a| < R$ and diverges for $|x - a| > R$.

The largest set of real numbers for which a power series converges is called the *interval of convergence* for the series. If the interval of convergence is finite, then it will always be centered on the number $x = a$ that the power series is centered on. The distance from $x = a$ to either endpoint of the interval of convergence is called the *radius of convergence.* The radius of convergence is the number R.

The proof of the theorem above relies on some axioms (assumptions we make) about the real numbers, so we will postpone the proof for now. Instead, let's take a look at some applications.

Example 17 *Consider the power series* $\sum_{j=0}^{\infty} 4^j (x-1)^j$. *What is the radius and interval of convergence?*

Solution. We constructed this power series in Example 10. In that example, we used the fact that the series is geometric to determine its interval of convergence. Let's use the method introduced in this section now to confirm our findings from Example 10. Observe

$$R = \lim_{j \longrightarrow +\infty} \frac{4^j}{4^{j+1}} = \lim_{j \longrightarrow +\infty} \frac{1}{4} = \frac{1}{4}$$

Based on this computation, the radius of convergence for the series will be $R = 1/4$, and the interval of convergence will be

$$-\frac{1}{4} + 1 < x < \frac{1}{4} + 1 \qquad \text{or} \qquad \frac{3}{4} < x < \frac{5}{4}$$

This is the same interval of convergence we determined in Example 10.

****

Example 18 *Consider the power series* $\sum_{j=1}^{\infty} \frac{(-1)^j}{(2j+1)!}(x-0)^{2j+1}$. *What is the radius and interval of convergence?*

Solution. First, observe that

$$A_j = \frac{(-1)^j}{(2j+1)!} \qquad \text{and} \qquad A_{j+1} = \frac{(-1)^{j+1}}{(2[j+1]+1)!} = \frac{(-1)^{j+1}}{(2j+3)!}$$

With this in mind, we let

$$\begin{aligned}
R &= \lim_{j \longrightarrow +\infty} \left| \frac{A_j}{A_{j+1}} \right| \\
&= \lim_{j \longrightarrow +\infty} \frac{1/(2j+1)!}{1/(2j+3)!} \\
&= \lim_{j \longrightarrow +\infty} \frac{(2j+3)!}{(2j+1)!} \\
&= \lim_{j \longrightarrow +\infty} \frac{(2j+3)\cdot(2j+2)\cdot(2j+1)!}{(2j+1)!} \\
&= \lim_{j \longrightarrow +\infty} (2j+3)\cdot(2j+2) \\
&= +\infty
\end{aligned}$$

We may conclude that this power series converges for all values of x. In this case, we say that the radius of convergence and interval of convergence are both infinite.

****

Problem 1. Determine the radius and interval of convergence for the power series $\sum_{j=0}^{\infty}(-1)^j\frac{(x-1)^j}{j+1}$.

Problem 2. Use the Ratio Test to determine the radius and interval of convergence for the power series $\sum_{j=0}^{\infty}(-1)^j\frac{x^{2j+1}}{2j+1}$.

Problem 3. Determine the radius and interval of convergence for the power series $\sum_{j=0}^{\infty}\frac{x^j}{j!}$. (Assume $0! = 1$.)

If a given power series is the Taylor series for a certain function f, we now have a way of determining the interval on which this series converges. Whether the series converges to the function f on this interval has yet to be proved. It turns out that this is *usually* the case.

At the moment, we have created Taylor series for several transcendental functions, and we have determined the intervals of convergence for these series:

1. $f(x) = e^x \longrightarrow \sum_{j=0}^{\infty} \frac{x^j}{j!}$ Series converges for all real values of x

2. $f(x) = \ln(x) \longrightarrow \sum_{j=0}^{\infty} (-1)^j \frac{(x-1)^j}{j+1}$ Series converges for $0 < x < 2$

3. $f(x) = \sin(x) \longrightarrow \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$ Series converges for all real values of x

4. $f(x) = \arctan(x) \longrightarrow \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j+1}$ Series converges for $-1 < x < 1$

Earlier in these notes, we gave some graphical motivation for the fact the series for $f(x) = \ln(x)$ and $f(x) = \arctan(x)$ actually converge to the functions they came from. In fact, *all* of these series converge on the specified intervals to the functions they were created from, but we will leave proof of this fact for another course.

ANSWERS

PART I

(1) $\sum_{j=0}^{\infty} (-1)^j \left(\frac{3}{4}\right)^j = \frac{4}{7}$ (2) $\sum_{j=0}^{\infty} \frac{1-2^j}{3^j} = -\frac{3}{2}$ (3) $\sum_{j=3}^{\infty} \frac{4^{j+1}}{5^j} = \frac{256}{25}$

(4) $\sum_{j=2}^{\infty} \frac{4^{2j}}{5^{2j-1}} = \frac{256}{45}$ (5) $\sum_{j=4}^{\infty} (-1)^j \frac{3-4^j}{6^{2j}} = -\frac{1169}{8631\,360}$ (6) $\sum_{j=0}^{\infty} (-1)^j \frac{2-2^{j+1}}{7^j} = \frac{7}{36}$

7. $f(x) = \sum_{j=0}^{\infty} 3(5x-3)^j$ Valid for $\frac{2}{5} < x < \frac{4}{5}$; $f(x) = \sum_{j=0}^{\infty} \frac{3 \cdot 5^j}{4^{j+1}} x^j$ Valid for $-\frac{4}{5} < x < \frac{4}{5}$

8. $f(x) = \sum_{j=0}^{\infty} 2x^{2j}$ Valid for $-1 < x < 1$

9. $f(x) = \sum_{j=0}^{\infty} (-1)^j \left(\frac{x}{2}\right)^{3j}$ Valid for $-2 < x < 2$

10. Observe that $0.121212... = \displaystyle\sum_{j=1}^{\infty} \frac{12}{100^j} = \frac{4}{33}$

11. Observe that $0.999... = \displaystyle\sum_{j=1}^{\infty} \frac{9}{10^j} = 1$

PART II

1. $T_4(x,1) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$

2. $T_5(x,0) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$; Observe that $e^1 \approx 2.718281828$ while
$$T_5(1,0) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} \approx 2.7167$$

3. $T_n(x,0) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + ... + \frac{1}{n!}x^n$

4. $T_6(x,0) = T_5(x,0) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

5. $T_6(x,0) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

6. $T_4(x,\pi/4) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{4}\right)^4$

7. $T_4(x,\pi/4) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4$
$$\tan\left(\frac{\pi}{3}\right) \approx 1.7321 \qquad \tan(0) = 0$$
$$T_4(\pi/3,\pi/4) = 1 + 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)^4 \approx 1.7242$$
$$T_4(0,\pi/4) = 1 + 2\left(0 - \frac{\pi}{4}\right) + 2\left(0 - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(0 - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(0 - \frac{\pi}{4}\right)^4 \approx 0.63932$$

8. $T_3(x,0) = x - \frac{2}{3}x^3 + \frac{2}{15}x^5$

9. $T_3(x,1/2) = \arcsin(x) = \frac{1}{6}\pi + \frac{2}{3}\sqrt{3}\left(x - \frac{1}{2}\right) + \frac{2}{9}\sqrt{3}\left(x - \frac{1}{2}\right)^2 + \frac{8}{27}\sqrt{3}\left(x - \frac{1}{2}\right)^3$

10. $T_6(x,1) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{30}(x-1)^6$