TRIGONOMETRY AND INTEGRATION

Trigonometry is a powerful tool for finding antideriative families of functions that involve sums or differences of squares. To see how, let's start by considering the indefinite integral

$$\int \sqrt{3-4x^2}dx$$

First, make the simple observation that

$$3-4x^2 = 3\left[1-\left(\frac{2x}{\sqrt{3}}\right)^2\right]$$

The first point of this observation is to note that the function $f(x)=\sqrt{3-4x^2}$ is only defined when

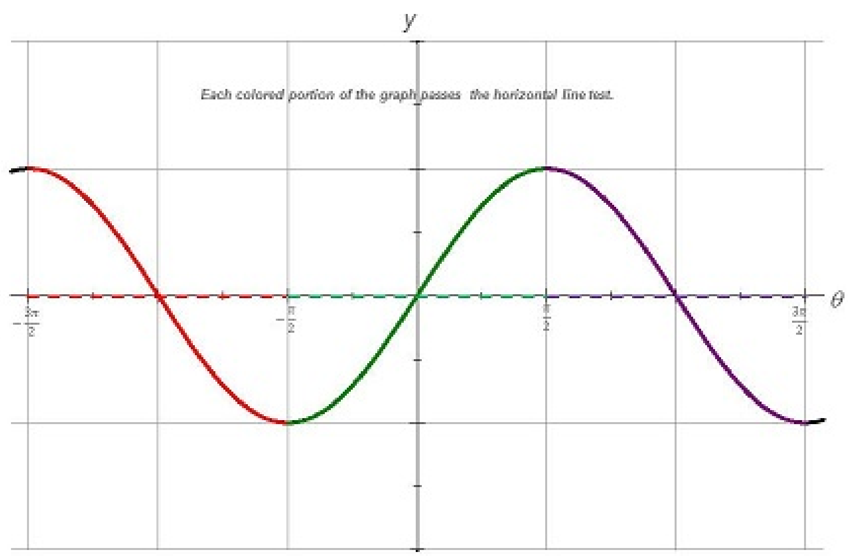
$$-\frac{\sqrt{3}}{2} \leq x \leq \frac{\sqrt{3}}{2}$$

The second point of this observation is to notice that what appears under the radical has the form "1 minus *something* squared." If we had a way to convert this expression into a perfect square *of some positive quantity*, we could remove the radical, thereby making the integral simpler.

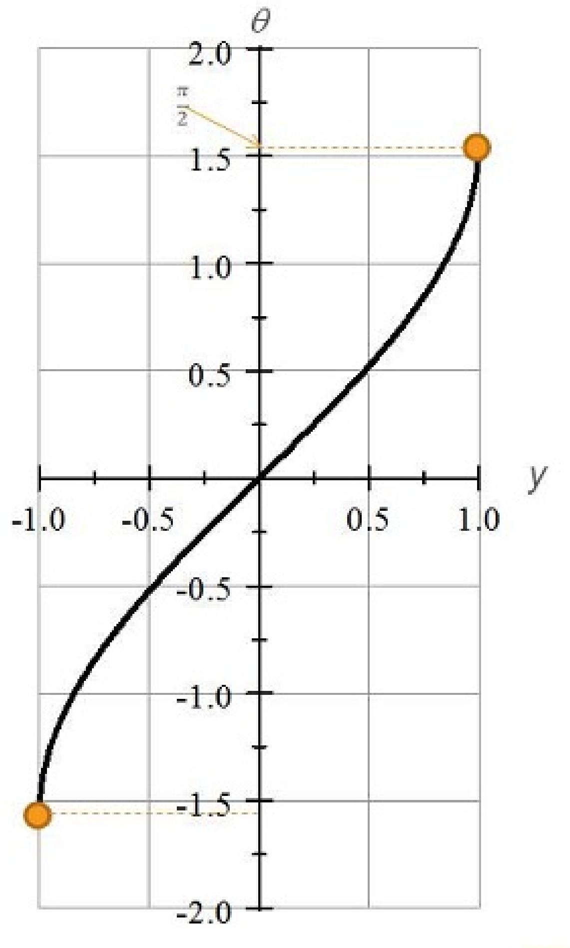
Now, here is where trigonometry comes into play. We know that for any angle whose radian measure θ , the Pythagorean identity

$$\cos^2(\theta) = 1-\sin^2(\theta)$$

is valid. The arcsine function is defined to be a *partial inverse* of the sine function. In particular, restrict attention to the graph of the sine function on the interval $-\pi/2 \leq \theta \leq \pi/2$ as shown below.



We could define a *partial inverse function* for the sine function on any one of these intervals. It is customary to use the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ to define a partial inverse. The partial inverse defined on this particular interval is called the *principal inverse sine* function; or more commonly, the *arcsine* function. We obtain the graph of the arcsine function by switching the roles of input and output variable (switching the horizontal and vertical axes) for the sine graph.



The function $\theta = g(y) = \arcsin(y)$ is defined by the graph shown above. This function reverses the sine function, but only on the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. In other words,

$$\arcsin(\sin(\theta)) = \theta$$

only when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

FUNDAMENTAL RELATIONSHIP BETWEEN SINE AND ARCSINE

Whenever $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we have $y = \sin(\theta)$ if and only if $\arcsin(y) = \theta$

Now, as long as we have $-\sqrt{3}/2 \leq x \leq \sqrt{3}/2$, we also have

$$-\frac{\pi}{2} = \arcsin(-1) \leq \arcsin\left(\frac{2x}{\sqrt{3}}\right) \leq \arcsin(1) = \frac{\pi}{2}$$

Therefore, it is possible to find a value of θ so that

$$\theta = \arcsin\left(\frac{2x}{\sqrt{3}}\right) \qquad \text{or equivalently} \qquad \frac{2x}{\sqrt{3}} = \sin(\theta)$$

Consequently, we also know that

$$3 - 4x^2 = 3\left[1 - \left(\frac{2x}{\sqrt{3}}\right)^2\right] = 3(1 - \sin^2(\theta)) = 3\cos^2(\theta)$$

Furthermore, we know that the cosine function is *nonnegative* as long its input lies between $-\pi/2$ and $\pi/2$. Therefore, we may conclude

$$\sqrt{3-4x^2} = \sqrt{3}\cos(\theta)$$

under the *trigonometric substitution* $\frac{2x}{\sqrt{3}} = \sin(\theta)$.

We can use this to our advantage when computing the indefinite integral above. Observe that the rules for implicit differentiation tell us

$$\begin{aligned} \frac{2x}{\sqrt{3}} = \sin(\theta) \quad &\Longrightarrow \quad \frac{d}{dx}\left[\frac{2x}{\sqrt{3}}\right] = \frac{d}{dx}\left[\sin(\theta)\right] \\ &\Longrightarrow \quad \frac{2}{\sqrt{3}} = \cos(\theta)\frac{d\theta}{dx} \end{aligned}$$

Therefore, we know

$$\begin{aligned} \int \sqrt{3-4x^2}dx &= \int \sqrt{3-4x^2}[1]dx \\ &= \int \sqrt{3}\cos(\theta)\left[\frac{\sqrt{3}}{2}\cos(\theta)\frac{d\theta}{dx}\right]dx \\ &= \frac{3}{2}\int \cos^2(\theta)d\theta \end{aligned}$$

By making an clever trigonometric substitution, we managed to eliminate the radical in the indefinite integral. However, doing so has now given an even power of the cosine function that we must antidifferentiate. Fortunately, trigonometry comes to the rescue.

POWER REDUCTION FORMULAS FOR SINE AND COSINE

$$\cos^2(\theta) = \frac{1}{2}\left[1+\cos(2\theta)\right] \qquad\qquad \sin^2(\theta) = \frac{1}{2}\left[1-\cos(2\theta)\right]$$

We can use the power reduction formula to convert the last indefinite integral into one that can be computed using only a simple substitution. Observe

$$\begin{aligned} \int \sqrt{3-4x^2}dx &= \frac{3}{2}\int \cos^2(\theta)d\theta \\ &= \frac{3}{4}\int \left[1+\cos(2\theta)\right]d\theta \\ &= \frac{3}{4}\left(\int 1d\theta + \int \cos(2\theta)d\theta\right) \qquad\qquad \text{Let } u = 2\theta \\ &= \frac{3}{4}\left(\int 1d\theta + \frac{1}{2}\int \cos(u)du\right) \\ &= \frac{3}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \end{aligned}$$

Of course, we are not yet done — we need to reverse the trigonometric substituition and recast the antiderivatives as functions of x. We know that

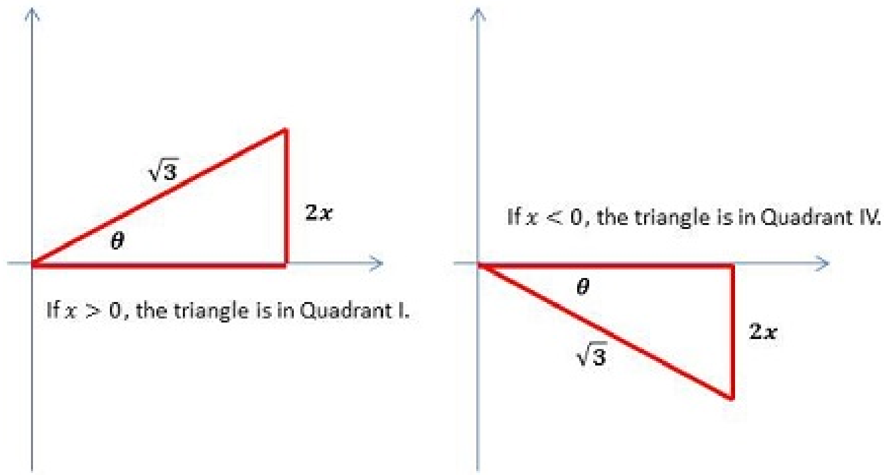
$$\theta = \arcsin\left(\frac{2x}{\sqrt{3}}\right)$$

but how do we handle the expression $\sin(2\theta)$? There are two steps we must use. First, we know that $-\pi/2 \le \theta \le \pi/2$. This tells us that θ is the radian measure of a vertex in a right triangle. (When this vertex is placed in standard position, the triangle will be in Quadrant I or Quadrant IV.)

Since θ is the measure of a vertex in a right triangle, we also know

$$\frac{\text{Side Opposite } \theta}{\text{Hypotenuse}} = \sin(\theta) = \frac{2x}{\sqrt{3}}$$

this suggests that we can let the hypotenuse of this triangle have length $\sqrt{3}$, and we can let the side opposite θ have (signed) length $2x$.



The Pythagorean Theorem therefore tells us

$$\text{Side Adjacent } \theta = \sqrt{\left(\sqrt{3}\right)^2 - (2x)^2} = \sqrt{3 - 4x^2}$$

Consequently, we know

$$\cos(\theta) = \frac{\text{Side adjacent } \theta}{\text{Hypotenuse}} = \frac{\sqrt{3-4x^2}}{\sqrt{3}}$$

Unfortunately, the expression $\sin(2\theta)$ appears in the antiderivatives instead of $\sin(\theta)$. However, trigonometry comes to the rescue once again.

DOUBLE ANGLE FORMULAS FOR SINE AND COSINE

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta) \qquad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

With all of this in mind, we finally see that

$$\begin{aligned}
\int \sqrt{3-4x^2}dx &= \frac{3}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{3}{4}\left[\arcsin\left(\frac{2x}{\sqrt{3}}\right) + \cos(\theta)\sin(\theta)\right] + C \\
&= \frac{3}{4}\left[\arcsin\left(\frac{2x}{\sqrt{3}}\right) + \left(\frac{\sqrt{3-4x^2}}{\sqrt{3}}\right)\left(\frac{2x}{\sqrt{3}}\right)\right] + C \\
&= \frac{3}{4}\left[\arcsin\left(\frac{2x}{\sqrt{3}}\right) + \frac{2x\sqrt{3-4x^2}}{3}\right] + C
\end{aligned}$$

Problem 1. Use the reduction formula for the sine function to find the antiderivative family for $f(x) = \sin^2(x)$.

Problem 2. Use trigonometric substitution to find the antiderivative family for $f(x) = \sqrt{4 - x^2}$.

Problem 3. Use the fact that $\cos^4(x) = \left[\cos^2(x)\right]^2$ along with the reduction formula for the cosine function to show that

$$\cos^4(x) = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

Problem 4. Find the antiderivative family for $f(x) = \cos^4(x)$.

Example 5. Use the method of trigonometric substitution to find the antiderivative family for $f(x) = x^2\sqrt{1-x^2}$.

Solution. In this case, we will let $\sin(\theta) = x$, so that $\theta = \arcsin(x)$. Implicit differentiation tells us that $1 = \cos(\theta)\frac{d\theta}{dx}$. Therefore, using the antiderivative family from Problem 4, we see

$$\begin{aligned}
\int x^2\sqrt{1-x^2}dx &= \int \sin^2(\theta)\sqrt{1-\sin^2(\theta)}\left[\cos(\theta)\frac{d\theta}{dx}\right]dx \\
&= \int \sin^2(\theta)\cos^2(\theta)d\theta \\
&= \int (1-\cos^2(\theta))\cos^2(\theta)d\theta \\
&= \int \cos^2(\theta)d\theta - \int \cos^4(\theta)d\theta \\
&= \frac{\theta}{2} + \frac{\sin(2\theta)}{4} - \left[\frac{\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32}\right] + C \\
&= \frac{3\theta}{8} - \frac{\sin(4\theta)}{32} + C
\end{aligned}$$

Once again, we resort to right triangles to reverse the trigonometric substitution. The angle whose radian measure is θ must be a vertex in a right triangle. Since we know that $\sin(\theta) = x$, we also know

$$\frac{\text{Side Opposite } \theta}{\text{Hypotenuse}} = \sin(\theta) = \frac{x}{1}$$

The Pythagorean Theorem tells us the side adjacent to θ has length $\sqrt{1-x^2}$. Therefore, we know

$$\cos(\theta) = \frac{\text{Side adjacent } \theta}{\text{Hypotenuse}} = \sqrt{1-x^2}$$

We can use the double angle formula for the sine and cosine functions to rewrite the antiderivative family in terms of $\sin(\theta)$ and $\cos(\theta)$ and then reverse the trigonometric substitution. Observe that

$$\begin{aligned}
\sin(4\theta) &= 2\cos(2\theta)\sin(2\theta) \\
&= 2\left[\cos^2(\theta) - \sin^2(\theta\right](2\sin(\theta)\cos(\theta)) \\
&= 4\left[\cos^3(\theta)\sin(\theta) - \sin^3(\theta)\cos(\theta)\right]
\end{aligned}$$

$$\begin{aligned}
\int x^2\sqrt{1-x^2}dx &= \frac{3\theta}{8} - \frac{\cos^3(\theta)\sin(\theta) - \sin^3(\theta)\cos(\theta)}{8} + C \\
&= \frac{3}{8}\arcsin(x) - \frac{1}{8}\left[x\left(1-x^2\right)^{3/2} - x^3\sqrt{1-x^2}\right] + C
\end{aligned}$$

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Problem 6. Use the fact that $\sin^3(\theta) = \sin(\theta)(1-\cos^2(\theta))$ to help you find the antiderivative family for $f(\theta) = \sin^3(\theta)$.

Problem 7. Use Problem 6 and trigonometric substitution to find the antiderivative family for $g(x) = \dfrac{x^3}{\sqrt{1-x^2}}$.

Now let's consider the indefinite integral

$$\int \sqrt{1+4x^2}dx$$

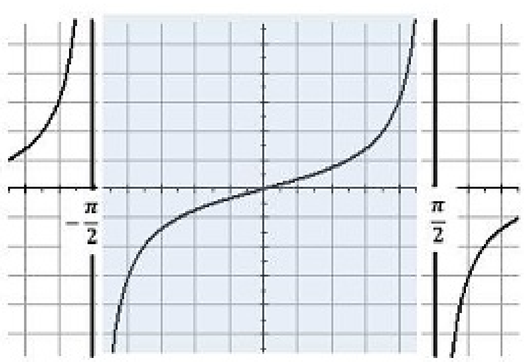
A trigonometric substitution is also possible for this integral — one that uses the tangent function instead of the sine function. First, observe that

$$1+4x^2 = 1+(2x)^2$$

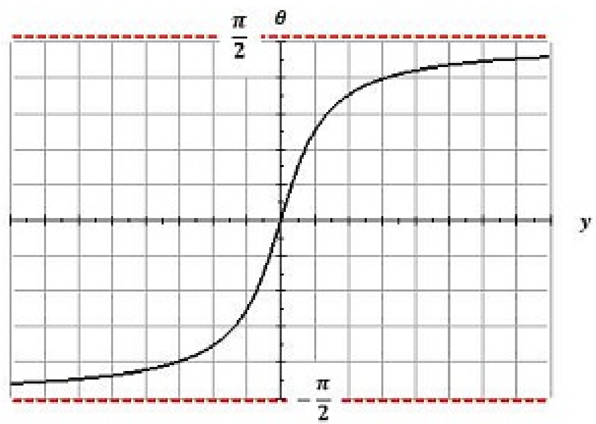
The point of this observation is to notice that what appears under the radical has the form "1 plus *something* squared." This pattern should remind you of the Pythagorean identity that relates the secant and tangent functions. We know that for any angle whose radian measure θ is not an odd multiple of $\pi/2$, the Pythagorean identity

$$\sec^2(\theta) = 1+\tan^2(\theta)$$

is valid. The arctangent function is defined to be a *partial inverse* of the tangent function. In particular, restrict attention to the graph of the tangent function on the interval $-\pi/2 < \theta < \pi/2$ as shown below.



The arctangent function is defined by forming the inverse function for the tangent function on this interval. The graph of the arctangent function is obtained by switching the input and output variables for the tangent function on this interval.



FUNDAMENTAL RELATIONSHIP BETWEEN TANGENT AND ARCTANGENT

Whenever $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have $y = \tan(\theta)$ if and only if $\arctan(y) = \theta$

Since the domain of the arctangent function is all real numbers, *for any value of* x, we can find a value of θ in the open interval $-\pi/2 < \theta < \pi/2$ so that $\theta = \arctan(2x)$. This equation is equivalent to the equation $2x = \tan(\theta)$. With this in mind, observe

$$1 + 4x^2 = 1 + (2x)^2 = 1 + \tan^2(\theta) = \sec^2(\theta)$$

Furthermore, the secant function *has positive output* for all values of θ such that $-\pi/2 < \theta < \pi/2$. Consequently, we know

$$\sqrt{1 + 4x^2} = \sec(\theta)$$

as long we we assume $\theta = \arctan(2x)$. Furthermore, since we have assumed $2x = \tan(\theta)$, the rules for implicit differentiation tell us

$$\frac{d}{dx}[2x] = \frac{d}{dx}[\tan(\theta)] \Longrightarrow 2 = \sec^2(\theta)\frac{d\theta}{dx} \Longrightarrow 1 = \frac{\sec^2(\theta)}{2}\frac{d\theta}{dx}$$

With all of this in mind, we can transform the indefinite integral. Observe

$$\begin{aligned} \int \sqrt{1 + 4x^2}\,[1]\,dx &= \int \sqrt{1 + \tan^2(\theta)}\left[\frac{\sec^2(\theta)}{2}\frac{d\theta}{dx}\right] dx \\ &= \frac{1}{2}\int \sec(\theta) \cdot \sec^2(\theta) d\theta \\ &= \frac{1}{2}\int \sec^3(\theta) d\theta \end{aligned}$$

Of course, this process is only useful if we can find the antiderivative family for the cube of the secant function.

Example 8. Use integration by parts to find the antiderivative family for $f(\theta) = \sec^3(\theta)$.

Solution. If we let $u = \sec(\theta)$, and let $dv = \sec^2(\theta)d\theta$, then we know that $du = \sec(\theta)\tan(\theta)d\theta$, and

$v = \tan(\theta)$. With this in mind, observe

$$\begin{aligned}
\int \sec^3(\theta)d\theta &= \int \sec^2(\theta)\sec(\theta)d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)\tan^2(\theta)d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec(\theta)\left[\sec^2(\theta) - 1\right]d\theta \\
&= \sec(\theta)\tan(\theta) - \int \sec^3(\theta)d\theta + \int \sec(\theta)d\theta
\end{aligned}$$

Notice that using integration by parts (along with the Pythagorean identity relating the tangent and secant functions), we have created a circular integral. In particular, we see that

$$2\int \sec^3(\theta)d\theta = \sec(\theta)\tan(\theta) + \int \sec(\theta)d\theta$$

Therefore, if we can find the antiderivative family for $g(\theta) = \sec(\theta)$, we will be done. There are several ways we could go about this, but probably the most direct is simple verification.

- If $G(\theta) = \ln|\sec(\theta) + \tan(\theta)|$, then $G'(\theta) = \sec(\theta)$.

  To see that this claim is true, we simply differentiate. Let $u = \sec(\theta) + \tan(\theta)$. Then

$$\begin{aligned}
\frac{d}{d\theta}\left[\ln|\sec(\theta) + \tan(\theta)|\right] &= \frac{d}{du}\left[\ln|u|\right]\frac{d}{d\theta}\left[\sec(\theta) + \tan(\theta)\right] \\
&= \left(\frac{1}{u}\right)\left[\sec(\theta)\tan(\theta) + \sec^2(\theta)\right] \\
&= \frac{\sec(\theta)\left[\tan(\theta) + \sec(\theta)\right]}{\sec(\theta) + \tan(\theta)} \\
&= \sec(\theta)
\end{aligned}$$

  Now, putting all of the pieces together, we see that

$$\int \sec^3(\theta)d\theta = \frac{1}{2}\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right] + C$$

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At long last, we can determine the antiderivative family for the function $f(x) = \sqrt{1+4x^2}$. Observe

$$\begin{aligned}
\int \sqrt{1+4x^2}dx &= \frac{1}{2}\int \sec^3(\theta)d\theta \\
&= \frac{1}{4}\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right] + C
\end{aligned}$$

Once again, we resort to right triangles to reverse the trigonometric substitutuion. We know that $\tan(\theta) = 2x$. Now, by assumption, $-\pi/2 < \theta < \pi/2$. This tells us that θ is a vertex angle in a right triangle (one that lies in Quadrant I or Quadrant IV when θ is in standard position). Furthermore,

$$\tan(\theta) = 2x \Longrightarrow \tan(\theta) = \frac{2x}{1} = \frac{\text{Side opposite } \theta}{\text{Side adjacent } \theta}$$

Consequently, the hypotenuse of this right triangle has length $\sqrt{1^2 + (2x)^2} = \sqrt{1+4x^2}$. Therefore,

$$\sec(\theta) = \frac{\text{Hypotenuse}}{\text{Side adjacent } \theta} = \sqrt{1+4x^2}$$

With all of this in mind, we see that

$$\begin{aligned}\int \sqrt{1+4x^2}dx &= \frac{1}{4}\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right] + C \\ &= \frac{1}{4}\left[2x\sqrt{1+4x^2} + \ln\left|\sqrt{1+4x^2}+2x\right|\right] + C\end{aligned}$$

Problem 9. Find the antiderivative family for $f(\theta) = \tan(\theta)\sec(\theta)$ by first converting the formula for f into one that involves sine and cosine functions.

Problem 10. Use Problem 9 and trigonometric substitution to find the antiderivative family for $f(x) = \dfrac{x}{\sqrt{1+x^2}}$.

HOMEWORK:

1. Use the fact that $\tan^2(\theta) = \sec^2(\theta) - 1$ to find the antderivative family for $g(\theta) = \tan^2(\theta)$.
2. Find the antiderivative family for $f(\theta) = \sin^4(\theta)$ and write your answer using only terms involving $\sin(\theta)$ and $\cos(\theta)$.
3. Use trigonometric substitution to find the antiderivative family for $f(x) = \dfrac{x^2}{\sqrt{9-x^2}}$.
4. Use the method from Problem 6 in the notes to help you find the antiderivative family for the function $g(\theta) = \sin^3(\theta)\cos^2(\theta)$.
5. Use the previous problem and trigonometric substitution to find the antiderivative family for $f(x) = x^3\sqrt{4-9x^2}$.

6. Use the fact that $\frac{d}{d\theta}[\sec(\theta)] = \sec(\theta)\tan(\theta)$ to help you find the antiderivative family for $g(\theta) = \sec^3(\theta)\tan(\theta)$.

7. Use the previous problem and trigonometric substitution to find the antiderivative family for $f(x) = x\sqrt{1+9x^2}$.

8. Use the Pythagorean identities to show that $\tan^2(\theta)\sec^3(\theta) = \sec^5(\theta) - \sec^3(\theta)$.

9. Use the previous problem along with integration by parts to find the antiderivative family for $g(\theta) = \sec^5(\theta)$.

10. Use the previous two problems and trigonometric substitution to find the antiderivative family for $f(x) = x^2\sqrt{1+x^2}$.

ANSWERS

1. $\displaystyle\int \tan^2(\theta)d\theta = \tan\theta - \theta + C$

2. $\displaystyle\int \sin^4(\theta)d\theta = \frac{3}{8}\theta - \frac{1}{2}\sin\theta\cos\theta + \frac{1}{8}(\cos^3\theta\sin\theta - \sin^3\theta\cos\theta) + C$

3. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}dx = \frac{1}{2}\left[9\arcsin\left(\frac{x}{3}\right) - x\sqrt{9-x^2}\right] + C$

4. $\displaystyle\int \sin^3(\theta)\cos^2(\theta)d\theta = \frac{\cos^5(\theta)}{5} - \frac{\cos^3(\theta)}{3} + C$

5. $\displaystyle\int x^3\sqrt{4-9x^2}dx = \frac{4}{3}\left[\frac{\left(4-9x^2\right)^{5/2}}{160} - \frac{\left(4-9x^2\right)^{3/2}}{24}\right] + C$

6. $\displaystyle\int \sec^3(\theta)\tan(\theta)d\theta = \frac{\sec^3\theta}{3} + C$

7. $\displaystyle\int x\sqrt{1+9x^2}dx = \frac{\left(1+9x^2\right)^{3/2}}{27} + C$

8. $\tan^2(\theta)\sec^3(\theta) = \left(\sec^2(\theta) - 1\right)\sec^3(\theta)$

9. $\displaystyle\int \sec^5(\theta)d\theta = \frac{\sec^3\theta\tan\theta}{4} + \frac{3}{8}\left[\sec\theta\tan\theta + \ln|\sec(\theta) + \tan(\theta)|\right] + C$

10. $\displaystyle\int x^2\sqrt{1+x^2}dx = \frac{x\left(1+x^2\right)^{3/2}}{4} - \frac{x\sqrt{1+x^2}}{8} - \frac{1}{8}\ln\left|x + \sqrt{1+x^2}\right| + C$