Improper Integrals

Recall that the notion of definite integral arose from the desire to compute the "net area" between the graph of a function f and the x-axis. The concept of "net area" between the graph of a function and the x-axis only makes sense under two restrictions:

- We only consider the graph of f on a closed and bounded interval $a \le x \le b$.
- We only consider functions which have finitely many jumps and no vertical asymptotes.

These restrictions were built into our definition of the definite integral, and we have assumed them ever since. However, in physics, engineering, and statistics, it is often necessary to consider the "net area" between the graph of a function f and the x-axis when one or both of these conditions does not hold. Consequently, it has become necessary to extend the notion of definite integral to cover situations in which at least one of these conditions fail. This is done by resorting to limits.

Definition 1. Let a be a fixed real number and suppose that f is a function that is defined for all real numbers $h > a$ or $h < a$. If it is the case that

$$\int_a^h f(x)\,dx$$

exists for all real numbers h, then we define the *improper* definite integrals

$$\int_a^{+\infty} f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^a f(x)\,dx$$

using limits. In particular,

$$\int_a^{+\infty} f(x)\,dx = \lim_{h\to+\infty}\int_a^h f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^a f(x)\,dx = \lim_{h\to-\infty}\int_h^a f(x)\,dx$$

When the limit exists, we say that the improper definite integral *converges*. When the limit is infinite, we say that the improper definite integral *diverges*.

Example 2. Compute $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$.

Solution. First, observe that for any real number $h \ge 1$, we have

$$\int_1^h \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^h = 1 - \frac{1}{h}$$

Armed with this information, we can see

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x^2}\,dx &= \lim_{h\to+\infty}\int_1^h \frac{1}{x^2}\,dx \\ &= \lim_{h\to+\infty}\left(1 - \frac{1}{h}\right) \\ &= 1\end{aligned}$$

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For Example 2, we would say that the definite integral *converges to* 1. We do not really think of this fact as meaning "the net area between the graph of $f(x) = 1/x$ and the x-axis is 1" on the ray $(1, +\infty)$ since net area does not make sense on an unbounded region.

Problem 1. Determine whether $\displaystyle\int_1^{+\infty} \frac{x}{(1+x^2)^{3/2}}\,dx$ converges or diverges.

Problem 2. Determine whether $\displaystyle\int_{-\infty}^{0} \frac{x}{\sqrt{1+x^2}}\,dx$ converges.

Problem 3. Determine whether $\displaystyle\int_1^{+\infty} x^{-1}\,dx$ converges.

Example 3. Show that $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$ converges to π.

Solution. First, we note that

$$\int \frac{1}{1+x^2}\,dx = \text{Arctan}(x) + C$$

Now, we deal with a "doubly infinite" improper definite integral in a simple way. First, we choose a convenient real number and spilt the integral into two limits.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx &= \int_{-\infty}^{1} \frac{1}{1+x^2}\,dx + \int_{1}^{+\infty} \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to-\infty} \int_{a}^{1} \frac{1}{1+x^2}\,dx + \lim_{h\to+\infty} \int_{1}^{h} \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to-\infty} \operatorname{Arctan}(x)\Big|_{a}^{1} + \lim_{h\to+\infty} \operatorname{Arctan}(x)\Big|_{1}^{h} \\
&= \lim_{a\to-\infty} (\operatorname{Arctan}(1) - \operatorname{Arctan}(a)) + \lim_{h\to+\infty} (\operatorname{Arctan}(h) - \operatorname{Arctan}(1)) \\
&= \lim_{h\to+\infty} \operatorname{Arctan}(h) - \lim_{a\to-\infty} \operatorname{Arctan}(a)
\end{aligned}$$

Now, we need to compute the values of these two limits. For this, we fall back on the way the inverse tangent function is defined. The inverse tangent function is defined by taking that portion of the tangent graph between $x = -\pi/2$ and $x = \pi/2$ and reflecting it about the line $y = x$. The tangent function has vertical asymptotes at both endpoints of this interval, so the inverse tangent will have horizontal asymptotes at $y = -\pi/2$ and $y = \pi/2$. In particular,

$$\lim_{a\to-\infty} \operatorname{Arctan}(a) = -\frac{\pi}{2} \qquad \text{and} \qquad \lim_{h\to+\infty} \operatorname{Arctan}(h) = \frac{\pi}{2}$$

Therefore,

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{h\to+\infty} \operatorname{Arctan}(h) - \lim_{a\to-\infty} \operatorname{Arctan}(a) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

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Example 4. Show that $\displaystyle\int_{1}^{+\infty} \sqrt{x}e^{-x}\,dx$ converges.

Solution. There is no easy way to compute the antiderivative family for $f(x) = \sqrt{x}e^{-x}$. However, there is a trick we can use to show that this improper definite integral does converge. Observe that, as long as $1 \le x$, we know $\sqrt{x} \le x$. Consequently, for $1 \le x$, we have $\sqrt{x}e^{-x} \le xe^{-x}$. Therefore,

$$\int_{1}^{h} \sqrt{x}e^{-x}\,dx \le \int_{1}^{h} xe^{-x}\,dx$$

This observation is helpful, because it *is* fairly easy to compute the value of the right-hand integral. First, note that

$$\begin{aligned}
\int xe^{-x}\,dx &= uv - \int v\,du \qquad \left\{\begin{array}{l} u = x \\ dv = e^{-x}\,dx \end{array}\right. \Longrightarrow \left\{\begin{array}{l} du = 1\,dx \\ v = -e^{-x} \end{array}\right. \\
&= -xe^{-x} + \int e^{-x}\,dx \\
&= -xe^{-x} - e^{-x} + C \\
&= -\frac{x+1}{e^x} + C
\end{aligned}$$

Consequently, we know that

$$\int_{1}^{h} xe^{-x}\,dx = -\frac{x+1}{e^x}\Big|_{1}^{h} = \frac{2}{e} - \frac{h+1}{e^h}$$

This tells us that

$$\int_{1}^{t} \sqrt{x}e^{-x}\,dx \le \frac{2}{e} - \frac{h+1}{e^h}$$

Therefore, we know that

$$\begin{aligned}\int_1^{+\infty} \sqrt{x}e^{-x}\,dx &= \lim_{h\to+\infty}\int_1^h \sqrt{x}e^{-x}\,dx \\ &\le \lim_{h\to+\infty}\left(\frac{2}{e}-\frac{h+1}{e^h}\right) \\ &= \frac{2}{e}\end{aligned}$$

since L'Hopital's Rule tells us

$$\lim_{h\to+\infty}\frac{h+1}{e^h} \underset{\text{LHR}}{=} \lim_{h\to+\infty}\frac{1}{e^h} = 0$$

Thus, we know that

$$\int_1^{+\infty} \sqrt{x}e^{-x}\,dx \le \frac{2}{e} < +\infty$$

and we may therefore conclude that the improper definite integral in question converges. (We do not, however, yet know what it converges to — we only know it converges to *something* no larger than $2/e$.)

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The previous example is an application of the *comparison test* for improper definite integrals. The comparison test can sometimes be used to show that an improper integral converges, but it cannot show what the integral converges to.

Theorem 5. (Comparison Test) Let a be a fixed real number and suppose that $f(h) \le g(h)$ for all $a < h$. Then the following statements are true:

(1) If $\displaystyle\int_a^{+\infty} f(x)\,dx$ diverges, then $\displaystyle\int_a^{+\infty} g(x)\,dx$ diverges.

(2) If $\displaystyle\int_a^{+\infty} g(x)\,dx$ converges, then $\displaystyle\int_a^{+\infty} f(x)\,dx$ converges.

Example 6. Show that $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all fixed p such that $0 < p \le 1$.

Solution. First, note that when $p = 1$, we have

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x}\,dx &= \lim_{h\to+\infty}\int_1^h \frac{1}{x}\,dx \\ &= \lim_{h\to+\infty} \ln|x|\Big|_1^h \\ &= \lim_{h\to+\infty} \ln(h) \\ &= +\infty\end{aligned}$$

Therefore, this improper definite integral diverges. Now, observe that if $0 < p < 1$, then $x^p \le x$ for all $1 < x$. Thus, $1/x \le 1/x^p$ for all $1 < x$; and we may conclude that

$$\int_1^{+\infty} \frac{1}{x^p}\,dx$$

diverges by the comparison test.

When using the comparison test, it is critical that the inequalities are "going the right way." For example, when $1 < x$, we know that $1/x^2 \leq 1/x$, so we also know that

$$\int_1^{+\infty} \frac{1}{x^2}\, dx \leq \int_1^{+\infty} \frac{1}{x}\, dx$$

However, the fact that the right-hand improper integral diverges DOES NOT imply that the left-hand improper integral diverges (we showed in Example 2 that it actually converges).

Problem 4. Use the comparison test to prove that $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\, dx$ converges for all fixed p such that $1 < p$.

Problem 5. Use the comparison test to prove that $\displaystyle\int_1^{+\infty} e^{-x^2}\, dx$ converges. (Hint: We know $x < x^2$ when $x > 1$.)

There is a second type of improper definite integral which arises when the function being considered has a vertical asymptote in the integration interval. This type is handled using limits as well.

Definition 7. Suppose that a function f is defined at all points in a closed and bounded interval $[a, b]$, except for a point $x = c$. Suppose further that f has a vertical asymptote at $x = c$. In this event, we let

$$\int_a^b f(x)\, dx = \lim_{h \to c^-} \int_a^h f(x)\, dx + \lim_{h \to c^+} \int_h^b f(x)\, dx$$

provided these limits exist. If the limits exist, we say the definite integral converges. If at least one limit is infinite, we say the definite integral diverges.

Example 8. Determine whether $\displaystyle\int_0^3 \frac{1}{\sqrt{x}}\, dx$ converges.

Solution. First, note that $f(x) = 1/\sqrt{x}$ has a vertical asymptote at $x = 0$. Therefore, the integral in question is improper. According to our definition,

$$\begin{aligned}
\int_0^3 \frac{1}{\sqrt{x}}\,dx &= \lim_{h\to 0^+} \int_h^3 x^{-1/2}\,dx \\
&= \lim_{h\to 0^+} 2\sqrt{x}\Big|_h^3 \\
&= 2\lim_{h\to 0^+} \left(\sqrt{3}-\sqrt{h}\right) \\
&= 2\sqrt{3}
\end{aligned}$$

Since the one-sided limit exists and is finite, we conclude the improper definite integral converges. Notice that only one limit was needed in this problem since the vertical asymptote lay at one endpoint of the integration interval.

Problem 6. Determine whether $\displaystyle\int_0^{10} \frac{1}{(x-2)^{2/3}}\,dx$ converges.

Example 9. Show that $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ diverges.

Solution. The function $f(x) = 1/x$ has a vertical asymptote at $x = 0$; hence the definite integral in question is improper. Observe that

$$\begin{aligned}
\int_0^1 \frac{1}{x^2}\,dx &= \lim_{h\to 0^+} \int_h^1 \frac{1}{x^2}\,dx \\
&= \lim_{h\to 0^+} \ln|x|\Big|_h^1 \\
&= \lim_{h\to 0^+} (\ln(1)-\ln(h)) \\
&= +\infty
\end{aligned}$$

since $F(h) = \ln(h)$ approaches $-\infty$ as h approaches 0 from the right. Hence, we conclude that this improper definite integral diverges.

Example 10. Show that $\displaystyle\int_{-1}^1 \frac{\ln(x^2)}{x}\,dx$ diverges.

Solution. First, note that we can find the antiderivative family for $f(x) = \ln(x^2)/x$ by first making the substitution $u = \ln(x^2)$ (so that the chain rule tells us $du = (2/x)\,dx$). Consequently,

$$\int \frac{\ln(x^2)}{x}\,dx = \frac{1}{2}\int u\,du = \frac{u^2}{4} + C = \frac{[\ln(x^2)]^2}{4} + C$$

Now, the natural log function has a vertical asymptote at $x = 0$; hence the function f has a vertical asymptote at $x = 0$ as well. Therefore, we know that

$$
\begin{aligned}
\int_{-1}^{1} \frac{\ln(x^2)}{x}\,dx &= \lim_{h\to 0^-} \int_{-1}^{h} \frac{\ln(x^2)}{x}\,dx + \lim_{h\to 0^+} \int_{h}^{1} \frac{\ln(x^2)}{x}\,dx \\
&= \lim_{h\to 0^-} \left.\frac{[\ln(x^2)]^2}{4}\right|_{-1}^{h} + \lim_{h\to 0^+} \left.\frac{[\ln(x^2)]^2}{4}\right|_{h}^{1} \\
&= \frac{1}{4}\lim_{h\to 0^-} \left([\ln(1)]^2 - [\ln(h^2)]^2\right) + \frac{1}{4}\lim_{h\to 0^+} \left([\ln(h^2)]^2 - [\ln(1)]^2]^2\right) \\
&= \frac{1}{4}\left(\lim_{h\to 0^-} [\ln(h^2)]^2 - \lim_{h\to 0^+} [\ln(h^2)]^2\right)
\end{aligned}
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This improper definite integral diverges because at least one (in this case both) of the one-sided limits is infinite. It looks like these one-sided limits should "cancel out"; however, since the two limits represent independent processes, they must be computed separately. Therefore this idea of "canceling out" is not definable, since neither $+\infty$ nor $-\infty$ is a real number.

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HOMEWORK: Section 7.8, Pages 534-535 Problems 7, 11, 13, 15, 17, 22, 24, 29, 31, 33, 37, 38, 49, 50, 51, 62, 63